NAUTICAL SURVEYING.

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BY THE LATE

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Obituary.

Vice-Admiral P. F. Shortland.—We have to announce the death, on the 18th inst., at Plymouth, in his 73rd year, of Vice-Admiral Peter Frederick Shortland, LL. D., Barrister-at-Law, whose ancestors for many generations served in the Royal Navy. He was the son of Capt. Thomas George Shortland, Captain-Superintendent of Jamaica Dockyard ; and, born in 1815, entered the Service on 15th January, 1827. In 1829 he passed through the Naval College at Portsmouth after ten months' study, the ordinary time at that period being about two years, and on 4th December, 1834, was appointed sub-lieutenant. He assisted, while serving in the *Rattlesnake*, in 1836-37, in making the settlement of Melbourne and a survey of Port Phillip for the benefit of the colony—his first work of the kind—receiving the approbation of his superiors and great praise from the Governor of the Colony. With the permission of the Admiralty Mr. Shortland matriculated at the University of Cambridge, in 1838, entered Pembroke College, and in 1842 came out Seventh Wrangler, and was forthwith elected a Fellow of his College. Immediately after finishing his examination at Cambridge he joined the *Excellent*, 1st April, 1842, as mate, and received special promotion to the rank of lieutenant on account of his superior attainments, and was sent to the North American Survey under the command of Capt. W. F. Owen. He was appointed to the command of the *Columbia* on the same Survey in 1844, and when she was paid off on 20th January, 1848, he received a Board promotion to the rank of commander. He was, in 1849, placed in charge of the Bay of Fundy and Nova Scotia Survey, and on 1st Jan., 1859, was gazetted post-captain, receiving the thanks of the Admiralty on the completion of the Survey in 1865. In 1865 he was appointed to the command of the *Hydra*, employed in the Mediterranean Survey, and was shortly afterwards ordered to the East Indies to take ocean soundings between Aden and Bombay. When the *Hydra* was paid off in 1868, Capt. Shortland was requested by the Hydrographer to write "A Sounding Voyage of Her Majesty's ship *Hydra*," which was printed by the Admiralty for the benefit of the Service, and was highly esteemed both in England and in the United States of America as a most scientific and useful work. Capt. Belknap, of the United States Navy, writing in the *United States Quarterly*, referred to it as the most important and satisfactory exploration of the depths which was made in 1868, as Capt. Shortland appeared to have a genius for deep-sea work, and his report embodied much valuable information from which Americans derived a great deal of instruction. Capt. Shortland received a captain's good service pension in April, 1870, and was placed on the retired list by Mr. Childers on 21st November, 1870, on attaining the age of 55. After his retirement he received the degree of LL.D. from his University, and was called to the Bar by the Hon. Society of Lincoln's-Inn on 27th January, 1873. He was promoted rear-admiral on the retired list on 21st September, 1876, and vice-admiral on the same list on 3rd January, 1881. He married, in 1848, Emily, eldest daughter of Capt. Thomas Jones, 74th Regiment Highland Light Infantry.

CONTENTS.

NAUTICAL SURVEYING.

CHAPTER I.

GENERAL CONSIDERATIONS.

It will be useful in the first place to trace out generally the line of practice a surveyor should follow, so far as the circumstances of the survey he has to make will permit.

All observations should be arranged and well considered beforehand, to ensure their being taken under the best possible conditions as regards weather, temperature, and time; the instruments must be reliable and the best that can be procured; they must be used with the greatest care and accuracy in accordance with the principles that will be explained hereafter and the practice resulting therefrom.

Every survey requires a series of main or principal points or stations, at which observations must be made to determine their geodetic relations to each other, as well as the positions of the intermediate objects.

Positions suitable for astronomical observations, and separated from each other by convenient distances, must be carefully selected at or as near as possible to one of the main triangulation stations, and here observations for latitude, time, and bearing, must be made; besides, every opportunity of making observations for true bearing at any of the other stations of the main triangulation must be embraced.

Base lines should be carefully measured wherever good level ground of sufficient length can be found.

The main stations of the triangulation near to or at which the astronomical observations are taken will hereafter be called astronomical stations.

The main triangulation and base measures, combined with the true bearing observations, give the geodetic relations of

the astronomical stations; the astronomical observations give the differences of latitude and longitude between each pair of astronomical stations, from which their bearings from each other and their distances in nautical miles must be calculated, and compared with the geodetic bearing and distance of the corresponding stations. If no errors existed in the observations, and the ratio of the unit of base measure to a nautical mile were correctly assumed, the bearings and distances of each pair of astronomical stations from each other, determined by these two entirely different processes, would agree exactly. The differences that are usually found between them arise from errors of observations, and from the ratio which the unit of base measure bears to a nautical mile being inaccurately assumed.

From each pair of astronomical stations we therefore obtain two bearings, determined by distinct and different methods, generally differing from each other by a small angle, and we suppose that the accurate bearing must be intermediate between them, and differ from each by a quantity proportional to the size of the error which each method was respectively liable to; we have besides two distances—one in nautical miles, and the other in terms of the linear unit—the comparison of which gives the ratio which the linear unit bears to a nautical mile.

After the observations at the astronomical stations throughout the survey have all been made, the bearings of each of the astronomical stations from each other taken in following order determined, and the ratio which the unit of base measure bears to a nautical mile have been found for each pair of astronomical stations as above described, they must be brought together and meaned in proportion to their values.

The relative values of the results of different observations are estimated from the following considerations. Observations taken continuously, by the same observer, with the same instrument, are considered equally good; and the most probable value of the quantity sought is the arithmetic mean of the whole; the sum of the differences between each observation and the arithmetic mean of the whole, taken regardless of sign, and divided by the number of the observations, is the most probable size of the error made in one observation, which may be either positive or negative; the probable size of the error of the arithmetic mean of all the observations is the size of the probable error of one observation divided by the number of the observations in the set. Thus, if n and n' be the number of observations respectively in two sets, s and s' the sum of the differences between the arithmetic mean of each set and each of the respective observations composing it taken irrespective

of sign, then the probable size of the errors of the means of the respective sets will be expressed by $\frac{s}{n^2}$ and $\frac{s'}{n'^2}$, and their relative values for meaning are as $\frac{n^2}{s} : \frac{n'^2}{s'}$.

To determine the probable size of the error in the bearing calculated from the astronomical differences of latitude and longitude.

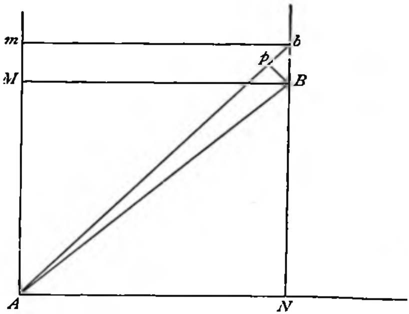


FIG. 1.

Let A and B (Fig. 1) be two astronomical stations, AMm the meridian through A and NBb the meridian through B; draw BM perpendicular to AMm, and let $Mm = m$ be the error in the difference of the latitudes observed at A and B; draw mb parallel to MB and cutting NBb in b; join Ab and AB and draw Bp perpendicular to AB, also let $AM = M$, $AB = D$, and $MB = N$.

Now BAb is the error in the bearing arising from the error m in the difference of latitude $= \frac{pB}{AB}$ very approximately $= \frac{Bb \times BM}{AB^2} = \frac{m \times N}{D^2}$. Similarly, if n is the error in the departure arising for the error in the astronomical difference of longitude,

Error in bearing arising therefrom $= \frac{n \times M}{D^2}$.

Hence if $\pm m$ be the probable size of the error in the observed difference of latitude M, and $\pm n$ be the probable size of the error in the departure arising from the probable size of the

error in the difference of longitude, then probable size of error in the bearing arising from the first will be $\pm\frac{m \times N}{D^2}$, and from the second $\pm\frac{n \times M}{D^2}$. Now these may have the same or different signs; if the same, their arithmetical values must be added; if of different signs, their arithmetical values must be subtracted from each other; and as each of these circumstances are equally probable, we must take their half sum as the probable size of the error resulting from two errors combined; consequently, the probable size of the error will be obtained by adding half the numerical sum of the probable sizes of the two errors to half their numerical difference, and on so doing the size of the error which is, numerically speaking, the larger, will be the probable size of the error resulting from the two errors combined, and $\therefore$ $\pm\frac{m \times N}{D^2}$ or $\pm\frac{n \times M}{D}$ will be the probable size of the error in the bearing according as $m \times N$ is $>$ or $<$ $n \times M$.

If B be the bearing of B (Fig. 1) from A, derived from the true bearing observations made at A and B and the main stations intermediate between them, combined with the triangulation, and B' the bearing calculated from the astronomical differences of latitude and longitude, $\pm p$ the probable size of the error of B, and $\pm p'$ that of B', then the most probable value of the bearing will be given by $\frac{p'B+pB'}{p+p'}$.

Having arranged all the astronomical stations in following order in pairs—the first pair being the first and second stations, and the second pair composed of the second and third stations, and so on, and calculated the bearings and distances for each pair of stations—it will generally be found that the direction of the meridian at the second astronomical station, derived from the bearings calculated from the observations made at and between the first and second astronomical stations, will differ slightly from that given by the observations made at and between the second and third astronomical stations. If β be the small angle between the two directions, $\pm p_1$ the probable size of the error in the first direction, and $\pm p_2$ that of the second, then the correction to be applied with opposite signs to the bearings derived from the first pair of stations will be $\frac{p_1\beta}{p_1+p_2}$, and to the second $\frac{p_2\beta}{p_1+p_2}$; so that the corrected direction will be intermediate between the other two,

and will be the most probable result that can be derived from the observations made at and between the three first astronomical stations of the series; and the probable size of the error in the corrected direction will be $\pm \frac{p_1 p_2}{p_1+p_2}$. If there were $n+1$ astronomical stations arranged in n pairs giving n directions to the meridians (when transferred to a common point) differing slightly from each other, the probable size of the error in the direction given by the first pair of astronomical stations being $\pm p_1$, its actual error x_1, the same elements corresponding to the second pair of stations being $\pm p_2$ and x_2 respectively, and so on to the nth pair of which $\pm p_n$ is the probable size of the error in the direction of the nth meridian and x_n its actual error; and if the small angle between the direction of the meridian given by the first pair of astronomical stations and that given by the second pair be denoted by β_1, that by the second and third pair β_2, and so on to that given by the $(n-1)$th and nth pair, which is represented in a similar manner by β_{n-1}, then we shall have

$$\begin{aligned} x_1 - x_2 &= \beta_1, \\ x_2 - x_3 &= \beta_2, \\ &\cdots\cdots\cdots \\ x_{n-1} - x_n &= \beta_{n-1}, \end{aligned}$$

where $\beta_1, \beta_2, \ldots \beta_{n-1}$ are all known. And if we suppose x_1, x_2, etc., as regards size, to be in proportion to p_1, p_2, etc., respectively, we may assume as probable that $\frac{x_1}{p_1}+\frac{x_2}{p_2}+\ldots+\frac{x_n}{p_n}=0$, and from these n equations determine the values of $x_1, x_2, \ldots x_n$.

Hence we have

$$\begin{aligned} x_1 - x_2 &= \beta_1 \ldots\ldots\ldots\ldots\ldots\ldots (1) \\ x_1 - x_3 &= \beta_1 + \beta_2 \ldots\ldots\ldots\ldots\ldots (2) \\ x_1 - x_4 &= \beta_1 + \beta_2 + \beta_3 \ldots\ldots\ldots\ldots (3) \\ x_1 - x_n &= \beta_1 + \beta_2 + \ldots + \beta_{n-1} \ldots\ldots (n-1) \end{aligned}$$

and $$\frac{x_1}{p_2}+\frac{x_2}{p_2}+\ldots+\frac{x_n}{p_n}=0 \ldots\ldots\ldots\ldots (n)$$

multiply the first equation by $\frac{1}{p_2}$, the second by $\frac{1}{p_3}$, ... and the $(n-1)$th by $\frac{1}{p_n}$, and, adding, we have

$$x_1\left\{\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}+\ldots+\frac{1}{p_n}\right\}-\left\{\frac{x_2}{p_2}+\frac{x_3}{p_3}+\ldots+\frac{x_n}{p_n}\right\}$$

$$=\beta_1\left\{\frac{1}{p_2}+\frac{1}{p_3}+\ldots+\frac{1}{p_n}\right\}+\beta_2\left\{\frac{1}{p_3}+\ldots+\frac{1}{p_n}\right\}+\ldots+\frac{\beta_{n-1}}{p_n}.$$

$$\therefore x_1\left\{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_n}\right\}-\left\{\frac{x_1}{p_1}+\frac{x_2}{p_2}+\ldots+\frac{x_n}{p_n}\right\}$$

$$=\beta_1\left\{\frac{1}{p_2}+\frac{1}{p_3}+\ldots+\frac{1}{p_n}\right\}+\beta_2\left\{\frac{1}{p_3}+\ldots+\frac{1}{p_n}\right\}+\ldots+\frac{\beta_{n-1}}{p_n}.$$

$\therefore$ equation (n) reduces this to

$$x_1=\frac{\beta_1\left\{\frac{1}{p_2}+\ldots+\frac{1}{p_n}\right\}}{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_n}}+\frac{\beta_2\left\{\frac{1}{p_3}+\ldots+\frac{1}{p_n}\right\}}{\frac{1}{p_1}+\ldots+\frac{1}{p_n}}+\ldots+\frac{\frac{\beta_{n-1}}{p_n}}{\frac{1}{p_1}+\ldots+\frac{1}{p_n}} \quad \text{(B)}$$

which gives x_1 in terms of the known quantities, and thence by means of equations (1), (2),...($n-1$), the values of $x_2 \ldots x_n$ immediately result, and being applied to their respective bearings will make them all consistent with each other, and give the most accurate direction of the meridians through the respective astronomical stations that can be obtained by giving to each and every observation its fair and proper effect.

If more than one base line is measured, the results must be compared as follows: Let l_1 and l_2 be the lengths of the same distance as given by the first and second base measures, and $\pm p_1$, $\pm p_2$ the probable sizes of their respective errors; let also $\mu_1=\frac{l_2-l_1}{2(l_1+l_2)}$; then the correction to be applied to the unit of the first base line measures will be $\frac{p_1\mu_1}{p_1+p_2}$, and to that of the second base line measures and the lengths depending on it, the correction $-\frac{p_2\mu_1}{p_1+p_2}$ must be applied. And if $n+1$ base lines are measured and a similar notation be adopted, we shall, by a process exactly similar to that explained for the bearings, find that the whole correction to be applied to the unit of the first base line measures will be

$$=\frac{\mu_1\left\{\frac{1}{p_2}+\ldots+\frac{1}{p_n}\right\}}{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_n}}+\frac{\mu_2\left\{\frac{1}{p_3}+\ldots+\frac{1}{p_n}\right\}}{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_n}}+\ldots+\frac{\frac{\mu_{n-1}}{p_n}}{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_n}} \quad \ldots\ldots\text{(M)}$$

and the correction to the others will immediately follow from this, and equations similar to (1), (2),...($n-1$).

To find the relation which the unit of base measure bears to a nautical mile in a given latitude.

Let q be the ratio which the unit of base measures bears to a mile of latitude in the middle latitude λ of the survey, q_1 that corresponding to latitude λ_1, the middle latitude of the

first pair of astronomical stations, D_1 the correct distance between the stations expressed in nautical miles, and $D_1+\delta_1$ that given by the astronomical differences of latitude and longitude derived from the observations made at them, δ_1 being the error arising from the errors of observation; we have the well-known relation:

$$\frac{q}{q_1}=\left\{\frac{1-\frac{\cos^2\lambda}{150}}{1-\frac{\cos^2\lambda_1}{150}}\right\}^{\frac{3}{2}}=r_1 \text{ suppose}\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore \qquad q=r_1q_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Let $\frac{q_1D_1}{D_1+\delta_1}=q_1{}' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$

$$\therefore \qquad q_1D_1=r_1q_1{}'\{D_1+\delta_1\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

In like manner the second pair of astronomical stations will give

$$qD_2=r_2q_2{}'\{D_2+\delta_2\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and so on to the last pair, which give

$$qD_n=r_nq_n{}'\{D_n+\delta_n\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(n+3).$$

To obtain the most probable value of q from these n different values, multiply each equation by its $D+\delta$, and add, when we shall have

$$q\{D_1(D_1+\delta_1)+D_2(D_2+\delta_2)+\ldots+D_n(D_n+\delta_n)\}$$
$$=q_1{}'r_1(D_1+\delta_1)^2+q_2{}'r_2(D_2+\delta_2)^2+\ldots+q_n{}'r_n(D_n+\delta_n)^2,$$

or, more conveniently written,

$$q\Sigma\{D(D+\delta)\}=\Sigma\{q'r(D+\delta)^2\} \ldots\ldots\ldots\ldots\ldots(n+4).$$

Add to each side of this equation $q\Sigma\{\delta(D+\delta)\}$,

$$\therefore \qquad q\Sigma\{(D+\delta)^2\}=\Sigma\{q'r(D+\delta)^2\}+q\Sigma\{\delta(D+\delta)\}$$

$$\therefore \qquad q=\frac{\Sigma\{q'r(D+\delta)^2\}}{\Sigma\{(D+\delta)^2\}}+q\frac{\Sigma\{\delta(D+\delta)\}}{\Sigma\{(D+\delta)^2\}}.$$

Let $\frac{\Sigma\{q'r(D+\delta)^2\}}{\Sigma\{(D+\delta)^2\}}=q''$

and $q\frac{\Sigma\{\delta(D+\delta)\}}{\Sigma\{(D+\delta)^2\}}=\eta q''$.

$$\therefore \qquad q=(1+\eta)q''\ldots\ldots\ldots\ldots\ldots\ldots(n+4),$$

since $D_1+\delta_1-\frac{D_1q}{q''}=D_1+\delta_1-\frac{q_1{}'r_1(D_1+\delta_1)}{q''}$

$$=(D_1+\delta_1)\left\{1-\frac{q_1{}'r_1}{q''}\right\};$$

substituting q by its value, given in $(n+4)$,

$$\delta_1-\eta D_1=(D_1+\delta_1)\left\{1-\frac{q_1{}'r_1}{q''}\right\}.$$

$$\therefore \quad (1+\eta)\frac{\delta_1}{D_1+\delta_1}-\eta=1-\frac{q_1'r_1}{q''};$$

$$\therefore \quad (1+\eta)\frac{\delta_1(D_1+\delta_1)}{(D_1+\delta_1)^2}-\eta=1-\frac{q_1'r_1}{q''}.$$

Similarly

$$(1+\eta)\frac{\delta_2(D_2+\delta_2)}{(D_2+\delta_2)^2}-\eta=1-\frac{q'_2r_2}{q''},$$

$$\ldots \quad = \quad \ldots$$

$$(1+\eta)\frac{\delta_n(D_n+\delta_n)}{(D_n+\delta_n)^2}-\eta=1-\frac{q_n'r_n}{q''};$$

adding these we have

$$(1+\eta)\Sigma\left\{\frac{\delta(D+\delta)}{(D+\delta)^2}\right\}-n\eta=n-\Sigma\left\{\frac{q'r}{q''}\right\}.$$

But $(1+\eta)\Sigma\left\{\frac{\delta(D+\delta)}{(D+\delta)^2}\right\}=\eta$;

$$\therefore \quad \eta(1-n)=n-\Sigma\left\{\frac{q'r}{q''}\right\}$$

$$\therefore \quad \eta=\frac{\Sigma\left\{\frac{q'r}{q''}\right\}-n}{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{E})$$

All the quantities in $\Sigma\left\{\frac{q'r}{q''}\right\}$ are known, and therefore its value can be calculated, and when substituted in equation (E) the value of η will result, and therefore $q=(1+\eta)q''$ becomes known also.

The most probable value of q that can be derived from all the observations having thus been found, those of $q_1, q_2, \ldots q_n$ will be found from the equations

$$q_1=\frac{q}{r_1},\ q_2=\frac{q}{r_2},\ \ldots\ q_n=\frac{q}{r_n},$$

q_1D_1, q_2D_2, ... q_nD_n having been found from the triangulation and base measures brought together as before pointed out, D_1, D_2, ... D_n become known also, and are the most probable values of the respective distances between the astronomical stations expressed in nautical miles that can be derived from all the observations fairly and properly combined.

The most probable values of the bearings and distances between the astronomical stations from each other having been thus established, calculate the differences of latitude, and apply them to correct the observed latitudes in the following manner:

Let λ_1+y_1, λ_2+y_2, ... λ_n+y_n be the correct latitudes of the astronomical stations whose latitudes determined from

the observations made at them for latitude are respectively $\lambda_1, \lambda_2, \ldots \lambda_n$, and let $\lambda_1 - \lambda_2 = M_1$, $\lambda_2 - \lambda_3 = M_2$, etc.

Let $M_1 + m_1$ be the corrected difference of latitude between the first two astronomical stations determined in the above manner. Then $y_1 - y_2 = m_1$. In like manner the second and third astronomical stations will give a similar equation, $y_2 - y_3 = m_2$, and so on for each pair, so that we shall have

$$\left.\begin{array}{llll} \text{from 3rd pair equation} & & & y_3 - y_4 = m_3 \\ \text{,,} \quad \text{4th} & \text{,,} & \text{,,} & y_4 - y_5 = m_4 \\ \quad \ldots & & & \quad \ldots \\ \text{,,} \quad n\text{th} & \text{,,} & \text{,,} & y_{n-1} - y_n = m_{n-1} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (\text{M})$$

Suppose the probable size of the error of $\lambda_1 = \pm p_1$, that of $\lambda_2 = \pm p_2$, and so on, then we may assume the most probable relation between the errors $y_1, y_2, \ldots y_n$ to be expressed by

$$\frac{y_1}{p_1} + \frac{y_2}{p_2} + \ldots + \frac{y_n}{p_n} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{N})$$

Adding the equations in (M) one after the other to their preceding sum we have

$$\begin{aligned} y_1 - y_2 &= m_1, \\ y_1 - y_3 &= m_1 + m_2, \\ y_1 - y_4 &= m_1 + m_2 + m_3, \\ \ldots &= \quad \ldots \\ y_1 - y_n &= m_1 + m_2 + \ldots + m_{n-1}; \end{aligned}$$

also $\qquad y_1 - y_1 = 0.$

Multiply the first of these by $\frac{1}{p_2}$, the second by $\frac{1}{p_3}$, ..., $n-1^{\text{th}}$ by $\frac{1}{p_n}$, and the last by $\frac{1}{p_1}$, and adding,

$$y_1 \left\{ \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_n} \right\} - \left\{ \frac{y_1}{p_1} + \frac{y_2}{p_2} + \ldots + \frac{y_n}{p_n} \right\}$$
$$= m_1 \left\{ \frac{1}{p_2} + \frac{1}{p_3} + \ldots + \frac{1}{p_n} \right\} + m_2 \left\{ \frac{1}{p_3} + \ldots + \frac{1}{p_n} \right\} + \ldots + \frac{m_{n-1}}{p_n}$$

which equation (N) reduces to

$$y_1 = \frac{m_1 \left\{ \frac{1}{p_2} + \frac{1}{p_3} + \ldots + \frac{1}{p_n} \right\} + m_2 \left\{ \frac{1}{p_3} + \ldots + \frac{1}{p_n} \right\} + \ldots + \frac{m_{n-1}}{p_n}}{\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{p_n}{1}}.$$

This gives the value of the correction to be applied to the observed latitude of the first astronomical station, and y_1 having been determined, the values of $y_2, y_3, \ldots y_n$ immediately result from the above equations; and by applying them to their respective observed latitudes, the most probable latitudes of the series of astronomical stations will be obtained.

The astronomical observations should be so arranged that the weather may have the least possible effect on them. When observations for latitude are required, the season of the year in which clear nights, even temperature, and calm weather usually occur must be selected; chronometrical meridian distances should be run when the temperature has its least variation and the rates of the chronometers have become regular and uniform. These are very important matters and must be carefully inquired into and acted upon. The observations to be made at each station must be well thought out beforehand and laid down on the principle that the heavenly bodies, their altitudes, and the number of observations on each, should be the same at all the stations, and should be made under similar circumstances, by the same observers, with the same instruments; and when making the observations, the observers must be in good health, and in a tranquil uniform state of mind and body; their instruments being carefully kept and handled so as to ensure, as far as possible, that they are in the same condition at each observation.

When about to commence a survey every possible information regarding the coast, climate, winds, and weather to be expected at different seasons of the year must be carefully collected, a preliminary examination of the coast being made in order to select the best positions for astronomical stations and for other purposes. These astronomical stations must be as near as possible to an anchorage where the vessel can anchor with safety, near a good and convenient landing place. The distances between the astronomical stations when chronometers are used to run the meridian distances will depend to a great extent on the speed of the vessel and the force at the surveyor's command, when he has a sufficient number of observers and chronometers at his disposal, so as to be able to land observers with chronometers at two astronomical stations. The distance between them should be about the number of miles the vessel is certain to cover in twenty-two hours, so that she can run backwards and forwards between the two stations and compare her chronometers with the chronometers at the stations at the same hour *exactly* every evening. If the surveyor has only sufficient means to observe at one station at a time, the distance the vessel can run in fourteen hours will point out the number of miles two consecutive stations can be placed from each other.

The instruments must be carefully examined before commencing the work, their character and value ascertained, and all excluded from primary observations that do not maintain their errors well. We will therefore in the next chapter show how this examination should be made.

CHAPTER II.

INSTRUMENTS.

WE will now proceed to examine a sextant.

First. Hold the sextant with its face upwards, clamp the index bar moderately, and push the vernier with a uniform gentle force slowly from one end of the arc to the other. If the resistance feels uniform, or not subject to any great variation, it shows that the arc of the sextant lies in one plane, to which the axis about which the index bar turns is perpendicular, or that these conditions are sufficiently satisfied for practical purposes; otherwise the instrument is not fit for use.

Secondly. Place the index at about 100°, and compare the arc seen directly with its reflection in the index glass, and if they appear in one unbroken line like the same arc, the plane of the index glass is perpendicular to the plane of the sextant; if this is not the case, the index glass must be brought into its proper position by the adjusting screws.

Thirdly. Place the index near zero, and view through the telescope the direct image of a star and that reflected from the horizon glass, clamp the index, and by means of the tangent screw make the reflected image slowly pass the direct image backwards and forwards. If the two images exactly coincide when passing, the horizon glass is parallel to the index glass; but if the reflected pass on one side of the direct image of the star, the coincidence when passing must be made exact by means of the adjusting screws.

Fourthly. Move the index so as to separate the direct and reflected images of a star by a distance nearly equal to the length of the parallel wires of the telescope, and turn the eyepiece carrying the parallel wires until, the direct image of the star coinciding with one extremity of the wire, the reflected image coincides with the other extremity; the wires will then be parallel to the plane of the sextant. This done, select two bright stars whose angular distance is about, but not less than, 120°, and bring the reflected image of one into coincidence

with the direct image of the other, making the coincidence first on the middle of one wire and then on the middle of the other. If the two readings agree, the axis of the telescope is parallel to the plane of reflection, and also, the previous adjustments having been made, to the plane of the sextant. If the two readings do not agree, the axis of the telescope is inclined to the plane of reflection, and must be made parallel thereto by turning the adjusting screws of the collar containing the telescope (supposed inverting), so as to make the axis of the telescope revolve in a direction from the wire on which the smaller reading was found towards that upon which the larger reading was obtained, until the two readings agree.

Fifthly. If the surfaces of the index glass are not parallel to each other, an error, which varies with the reading of the sextant, will result. This may be determined by measuring a large angle, not less than 120°, between two known stars, then taking out the index glass, turning it upside down, and replacing it in the frame and remeasuring the same angle; the sextant angles, being corrected for the known errors of the instrument, will be equally and oppositely affected by the inclination of the surfaces of the index glass to each other, and their difference, supposing the angular distance of the stars to have remained constant, will be twice the error corresponding to the reading of the sextant for the distance between the stars. As will be explained hereafter, a change in the temperature and pressure of the atmosphere will cause a small change in the apparent distance of the two stars, which must be calculated and applied to the result as above obtained should any such change occur, and the temperatures and pressures of the atmosphere at the times of the two observations differ from each other.

Sixthly. A want of parallelism in the surfaces of the horizon glass produces an error which is constant for all angles, and will therefore be included in the index error determined in the ordinary way.

Seventhly. There is another error which is not liable to change unless the instrument suffers some great and unusual derangement, as we will now explain.

Let $ABCG$ (Fig. 2) be the plane of a sextant, ABC its graduated arc, of which G is the centre, M the point in which the axis, about which the index bar turns, meets the plane of the graduated arc. The points G and M ought to be coincident; but as this agreement is seldom exact, it is necessary to determine the error introduced into the reading of an observed angle by these points not being coincident.

In a well made instrument the distance between G and M is

very small, and their relative situation does not change unless the sextant meets with a serious accident.

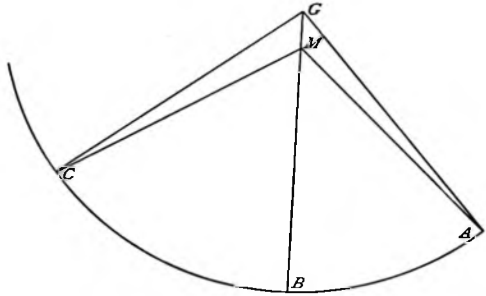


FIG. 2.

Let A be the zero of arc graduation, C the point on the arc where the reading is taken when a given angle A is observed; join GA, MA, GC, MC, and GM, which produce to meet the arc ABC in B. We will suppose the sextant free from other errors.

Let a = the reading of the arc at B;
θ' = " " " C;
$AG = r$, $GM = d$.

Since d is always very small with respect to r, the angles GAM and GCM will always be so small that their circular measures may be taken for their sines without introducing any practical error.

$$\therefore \qquad GAM = \frac{GM}{GA}\sin AMB = \frac{d}{r}\sin\frac{a}{2}.$$

Similarly $\quad GCM = \dfrac{d}{r}\sin\dfrac{\theta' - a}{2};$

putting $e = \dfrac{2d}{r\sin 1''}$ when these angles are expressed in seconds,

$$2GAM = e\sin\frac{a}{2},$$

$$2GCM = e\sin\frac{\theta' - a}{2}.$$

The observed angle $= 2AMC$

$$= 2\{AGC + GAM + GCM\}$$

$$= \theta' + e\left\{\sin\frac{\theta' - a}{2} + \sin\frac{a}{2}\right\}.$$

The point C is where the vernier division meeting that on the graduated arc gives the exact reading of the instrument, which we will denote by θ. Hence if A be the observed angle we shall have

$$A=\theta+e\left\{\sin\frac{\theta'-a}{2}+\sin\frac{a}{2}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Suppose now that the surfaces of the index and horizon glasses were not exactly parallel when the sextant reading was zero, but as is generally the case the index bar required to be moved through a small angle $\frac{\gamma}{2}$ in order to make an exact coincidence between the direct and reflected images of the same object: in this position let θ_0 be the reading of the sextant and θ_0' that where the coincidence of the vernier and are divisions gives the reading θ_0; substituting these values in (1)

$$\gamma=\theta_0+e\left\{\sin\frac{\theta_0'-a}{2}+\sin\frac{a}{2}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The reading θ now corresponds to an angle $A+\gamma$, and therefore

$$A+\gamma=\theta+e\left\{\sin\frac{\theta'-a}{2}+\sin\frac{a}{2}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Subtracting equation (2) from this,

$$A=\theta-\theta_0+e\left\{\sin\frac{\theta'-a}{2}-\sin\frac{\theta_0'-a}{2}\right\}\ldots\ldots\ldots(4)$$

$$=\theta-\theta_0+2e\left\{\sin\frac{\theta'-\theta_0'}{4}\cos\frac{\theta'+\theta_0'-2a}{4}\right\}\ldots(5)$$

$$=\theta-\theta_0+2e\sin\phi\,.\cos\left(\psi-\frac{a}{2}\right)\ldots\ldots\ldots\ldots\ldots(6)$$

where $\qquad \phi=\dfrac{\theta'-\theta_0'}{4}$ and $\psi=\dfrac{\theta'+\theta_0'}{4}$.

To find the values of e and a, measure the known angle between two stars whose distance is about 60°, and also that between two stars whose distance is somewhere between 120° and 130°; then if D_1 be the first distance, and θ_1, ϕ_1, and ψ_1 the corresponding values of θ, ϕ, and ψ: D_2 the second distance, and θ_2, ϕ_2, and ψ_2 the corresponding values of θ, ϕ, and ψ respectively, we shall have two equations:

$$\left.\begin{aligned}D_1&=\theta_1-\theta_0+2e\sin\phi_1\cos\left(\psi_1-\frac{a}{2}\right)\\ D_2&=\theta_2-\theta_0+2e\sin\phi_2\cos\left(\psi_2-\frac{a}{2}\right)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(\text{A})$$

from which the values of e and a can be determined in the following manner:

Putting $D_1 - \theta_1 + \theta_0 = 2d_1$, and $D_2 - \theta_2 + \theta_0 = 2d_2$, equations (A) become

$$\left.\begin{aligned} e \sin\phi_1 \cos\left(\psi_1 - \frac{\alpha}{2}\right) &= d_1 \text{ and} \\ e \sin\phi_2 \cos\left(\psi_2 - \frac{\alpha}{2}\right) &= d_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{B})$$

$$\therefore \qquad \cos\left(\psi_1 - \frac{\alpha}{2}\right) = \frac{d_1 \sin\phi_2}{d_2 \sin\phi_1} \cos\left(\psi_2 - \frac{\alpha}{2}\right),$$

$$= \beta \cos\left(\psi_2 - \frac{\alpha}{2}\right), \text{ if we put } \beta \text{ for } \frac{d_1 \sin\phi_2}{d_2 \sin\phi_1}.$$

Consequently

$$\cos\psi_1 \cos\frac{\alpha}{2} + \sin\psi_1 \sin\frac{\alpha}{2} = \beta \cos\psi_2 \cos\frac{\alpha}{2} + \beta \sin\psi_2 \sin\frac{\alpha}{2},$$

$$\therefore \qquad (\sin\psi_1 - \beta \sin\psi_2) \sin\frac{\alpha}{2} = (\beta \cos\psi_2 - \cos\psi_1) \cos\frac{\alpha}{2},$$

$$\therefore \qquad \tan\frac{\alpha}{2} = \frac{\beta \cos\psi_2 - \cos\psi_1}{\sin\psi_1 - \beta \sin\psi_2},$$

$$= \cot\psi_1 \frac{\beta \dfrac{\cos\psi_2}{\cos\psi_1} - 1}{1 - \beta \dfrac{\sin\psi_2}{\sin\psi_1}} \quad \ldots\ldots\ldots\ldots\ldots (\text{D})$$

To place this in a form adapted to logarithmetic computation. First, when $\beta \dfrac{\cos\psi_2}{\cos\psi_1}$ is positive and greater than unity, $\beta \dfrac{\sin\psi_2}{\sin\psi_1}$ must also be greater than unity; assume $\beta \dfrac{\cos\psi_2}{\cos\psi_1} = \sec^2 x$, and $\beta \dfrac{\sin\psi_2}{\sin\psi_1} = \sec^2 y$: then equation (D) becomes

$$\tan\frac{\alpha}{2} = -\cot\psi_1 \frac{\tan^2 x}{\tan^2 y},$$

the negative sign pointing out that $\dfrac{\alpha}{2}$ is intermediate in value between 90° and 180°.

Secondly, when $\beta \dfrac{\cos\psi_2}{\cos\psi_1}$ is positive and less than unity, whilst $\beta \dfrac{\sin\psi_2}{\sin\psi_1}$ is greater than unity, assume $\beta \dfrac{\cos\psi_2}{\cos\psi_1} = \cos^2 x$, and $\beta \dfrac{\sin\psi_2}{\sin\psi_1} = \sec^2 y$: equation (D) becomes $\tan\dfrac{\alpha}{2} = \cot\psi_1 \dfrac{\sin^2 x}{\tan^2 y}$, and $\dfrac{\alpha}{2}$ will lie between 0° and 90°.

Thirdly, when $\beta\dfrac{\cos\psi_2}{\cos\psi_1}$ and $\beta\dfrac{\sin\psi_2}{\sin\psi_1}$ are both positive and less than unity, assume the first $=\cos^2 x$, and the second $=\cos^2 y$, and equation (D) gives

$$\tan\frac{a}{2} = -\cot\psi_1\frac{\sin^2 x}{\sin^2 y}.$$

Lastly, when β is negative, assume $\beta\dfrac{\cos\psi_2}{\cos\psi_1} = -\tan^2 x$, and $\beta\dfrac{\sin\psi_2}{\sin\psi_1} = -\tan^2 y$; when equation (D) becomes

$$\tan\frac{a}{2} = -\cot\psi_1\frac{\sec^2 x}{\sec^2 y}.$$

Having thus calculated the value of $\frac{a}{2}$ from equation (B), select that in which the value of its d is numerically greater than the other, then, $\frac{a}{2}$ being substituted by its value in the term containing it, that of e can be easily calculated.

The adjustment mentioned fourth in order (p. 11) having been made, and the line of collimation of the telescope brought as nearly as possible parallel to the plane of the sextant by the adjusting screws of the telescope collar, they should not be touched, but the small error of collimation that may remain must be determined by observation in the following manner:

In the plane of the paper draw the straight lines OI, OH parallel to the normals to the surfaces of the index and horizon glasses of a sextant respectively (Fig. 3); and from O as a centre describe the surface of a sphere, cutting the plane of the paper in the great circle $AHE'IB$, of which P is the pole: draw the straight line OE parallel to the line of collimation of the telescope, and let it be inclined to the plane of the paper at a small angle x. Let SO be parallel to a ray from an object which after incidence on the index glass is reflected in a direction parallel to OR, so as to be incident on the horizon glass, and again reflected in a direction parallel to OE, so as to be seen through the telescope; also let these straight lines meet the spherical surface, of which O is the centre, in the points S and R; draw the great circles passing through P and the points S, R, and E, and produce them when necessary to meet the great circle $AHE'IB$ in the points B, A, and E' respectively. Since SO, OI, and OR are in the same plane, the circle SIR passing through the points S, I, and R is a great circle; for a similar reason the circle passing through R, H, and E is a great circle; join ES by the arc of a great circle.

Then $EE' = x$; let $IH = \theta$, and $E'H = T$, $RI = IS$, and $EH = HR$.

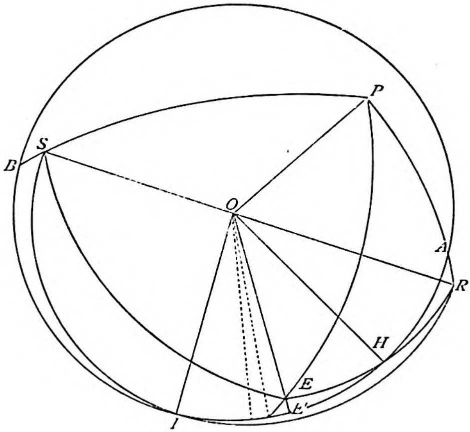


FIG. 3.

Since the triangles AIR, BIS have each a right angle, the angles at I equal to each other, and $RI = IS$, they are equal and similar to each other.

$\therefore$ $AR = BS$, and $AI = AB$; similarly $AR = EE'$;

$\therefore$ $BS = AR = EE' = x$;

also $AH = HE' = T$, and $IE' = IH - E'H = \theta - T$;

but $AI = AH + HI = T + \theta$;

$\therefore$ $BE' = BI + IE' = AI + I'E = 2\theta$.

Now observed angle $= ES$;

angle $SPE = BE' = 2\theta$; $PS = \frac{\pi}{2} - x$; $PE = \frac{\pi}{2} - x$;

therefore in triangle SPE

$\cos ES = \cos PS \cdot \cos PE + \sin PS \cdot \sin PE \cdot \cos SPE$

$= \sin^2 x + \cos^2 x \cos 2\theta$

$= \cos 2\theta + 2 \sin^2 x \sin^2 \theta$. Now x is so small that it may be substituted for its sine without introducing an error of any practical importance. Doing this, and changing circular measure into degrees and parts, we shall find

$$ES = 2\theta - x^2 \sin 1'' \tan \theta \quad \text{(E)}$$

where x is expressed in seconds, and θ in degrees, minutes, and seconds.

To find the value of x, place the parallel wires of the telescope perpendicular to the plane of the instrument, by turning the eye-piece containing the wires until the cross wires perpendicular to the parallel wires are parallel to the plane of the sextant, in the same way as before described for making the parallel wires parallel to the plane of the instrument. Bring the direct image of a bright star on the middle point of one of the parallel wires, and its reflected image on that of the other by moving the index bar on the arc, then change the wires, bringing the reflected image of the star to the wire on which the direct image was by moving the index bar off the arc, turning the instrument until the direct image is on the centre of the other wire; the readings of the sextant being carefully taken in both positions, viz. when the index bar was on the arc, and then when the contacts with the wires were made off the arc. Half the sum of the two readings will give the angular distance between the parallel wires; let this be $2d$ seconds of arc. Bring the parallel wires to their proper position, and measure the angular distance between two stars, which is not less than and does not greatly exceed 120°, first making the coincidence at the middle point of one of the parallel wires and then at the middle point of the other. Let θ_1 and θ_2 be the readings of the sextant, A the correct value of the angular distance between the stars, and x the error of collimation. Then if $x+d$ is the inclination of the line of sight to the plane of the sextant at the first observation, $x-d$ will be that for the second. Substituting these values for x in (E), and observing that θ the ordinary reading of the sextant differs but slightly from θ_1 or θ_2 and from $\frac{\theta_1+\theta_2}{2}$, it may be substituted for either in the term multiplied by $x^2 \sin 1''$ without introducing any practical difference in the results obtained from so using it:

$$\therefore \quad A=\theta_1-(x+d)^2 \sin 1'' \tan\frac{\theta}{2},$$

$$A=\theta_2-(x-d)^2 \sin 1'' \tan\frac{\theta}{2},$$

$$0=(\theta_1-\theta_2)-4dx \sin 1'' \tan\frac{\theta}{2}:$$

$$\therefore \quad x=\frac{\theta_1-\theta_2}{4d \sin 1'' \tan\frac{\theta}{2}}, \text{ where } \frac{\theta}{2}=\frac{\theta_1+\theta_2}{4}.$$

When the surfaces of the index and horizon glasses are respectively perpendicular to the plane of the sextant, the plane of reflection is parallel to the sextant and x is the collimation error; but this is not generally the case owing to small errors made in adjusting the glasses, the effect of which we must now examine.

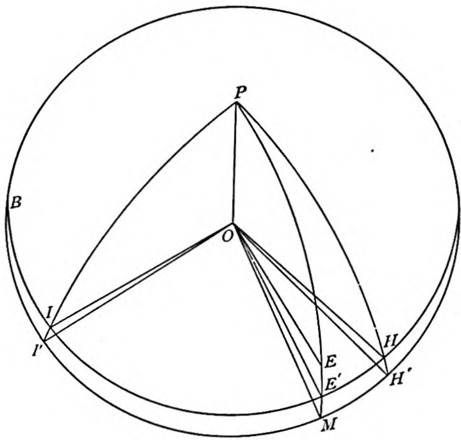


FIG. 4.

Suppose the plane of the paper to be parallel to the plane of the sextant, and to cut the sphere of which O (Fig. 4) is the centre in the great circle $AH'MI'B$ whose pole is at P; draw OI parallel to the normal to the surface of the index glass, and OH parallel to the normal to the surface of the horizon glass, the plane OIH cutting the surface of the sphere in the great circle $AHE'IB$ will be parallel to the plane of reflection; draw OE meeting the surface of the sphere in E parallel to the line of collimation of the telescope of the sextant, and let the great circles passing through P, and the points I, E, and H respectively, cut the great circle $AH'MI'B$ in the points I', M, and H'; and let the great circle IEM cut $AHE'IB$ in E'.

Let $HH'=h$; $II'=i$; $EM=c$; $EE'=x$; $E'M=y$; $I'H'=\theta$; $H'M=T$; $IAI'=\psi$; and $AH'=\phi$.

Then $AM=\phi+T$; and $AI'=\phi+\theta$.

The triangles AII', $AE'M$, and AHH' are right-angled at I', M, and H' respectively; therefore

$$\sin AH' = \tan HH' \cot HAH',$$

$$\therefore \quad \tan HH' = \sin AH' \tan HAH',$$

or $$\tan h = \sin \phi \tan \psi.$$

Similarly $$\tan y = \sin(\phi + T)\tan \psi$$

and $$\tan i = \sin(\phi + \theta)\tan \psi$$

$$= \tan h(\cos \theta + \cot \phi \sin \theta).$$

Similarly $$\tan y = \tan h\{\cos T + \cot \phi \sin T\},$$

$$\therefore \quad \sin \theta \tan y = \tan h\{\cos T \sin \theta - \sin T \cos \theta\} + \tan i \sin T;$$

$$\therefore \quad \tan y = \frac{\tan h \sin(\theta - T) + \tan i \sin T}{\sin \theta}.$$

This gives the value of y in terms of the errors h and i, the constant angle T, and the variable θ which represents half the reading of the sextant; h and i are always small in practice, and y also so long as the reading of the sextant is not very small; we may therefore put $y = \dfrac{h \sin(\theta - T) + i \sin T}{\sin \theta}$, unless θ is very small, without introducing any sensible error.

Therefore when x has been determined in the manner before mentioned we shall have $c = x + y$

$$= a + \frac{h \sin(\theta - T) + i \sin T}{\sin \theta} \text{..................(E)}$$

so that when h, i, and T have been determined, c, the collimation error, will be known.

The other errors depending on h and i may be determined as follows in the triangle IPH:

$$\cos IH = \cos PI \cos PH + \sin PI \sin PH \cos IPH;$$

$$= \sin i \sin h + \cos i \cos h \cos \theta;$$

$$= ih + \left(1 - \frac{i^2}{2} - \frac{h^2}{2}\right)\cos \theta$$

very approximately, since i and h are both very small.

$$\therefore \quad \cos IH = \cos \theta - \frac{(i-h)^2}{2} + (i^2 + h^2)\sin^2\frac{\theta}{2};$$

$$\therefore \quad IH = \theta + \frac{(i-h)^2 \sin 1''}{2 \sin \theta} - \frac{(i^2 + h^2)}{2} \sin 1'' \tan \frac{\theta}{2} \text{.......(G)}$$

where θ is expressed in degrees, minutes, and seconds, and i and h in seconds respectively.

By equation (E) the observed angle

$$A = 2IH - x^2 \sin 1'' \tan IH$$

$$= 2\theta + \frac{(i-h)^2 \sin 1''}{\sin \theta} - (i^2 + h^2)\sin 1'' \tan \frac{\theta}{2}$$

$$- \left\{c - \frac{h \sin(\theta - T) + i \sin T}{\sin \theta}\right\}^2 \sin 1'' \tan \theta \text{........(H)}$$

where 2θ is the reading of the instrument, and c, i, and h constant instrumental errors expressed in seconds. Combining this with equation (4), and taking θ for the reading of the instrument, and θ' that of the arc where the vernier and arc divisions coincide, we have generally

$$A=\theta-\theta_0+c\left\{\sin\frac{\theta'-a}{2}-\sin\frac{\theta_0'-a}{2}\right\}+\frac{(i-h)^2\sin 1''}{\sin\frac{\theta}{2}}-(i^2+h^2)\sin 1''\tan\frac{\theta}{4}$$

$$-\left\{c-\frac{h\sin\left(\frac{\theta}{2}-T\right)+i\sin T}{\sin\frac{\theta}{2}}\right\}^2\sin 1''\tan\frac{\theta}{2}\text{...........(8)}$$

The error in the position of the index glass of a well adjusted sextant ought not to exceed four minutes. The horizon glass can be adjusted to the index glass with considerable accuracy, so that when the adjustment is carefully made their surfaces ought not to be inclined to each other more than one minute of angle. Supposing these to be the limiting values of i and h, the term $\frac{(i-h)^2}{\sin\frac{\theta}{2}}\sin 1''$, unless θ is very small and less than 12′, may be neglected without introducing an error sufficiently large to be of any practical importance. To prove this let $(i-h)=\pm 1'$ and $\therefore (i-h)^2=60\times 60''$;

hence $$\frac{(i-h)^2}{\sin\frac{\theta}{2}}\sin 1''=60\times 60\sin 1''\operatorname{cosec}\frac{\theta}{2}.$$

log 360 - - - - - - -	2·5563
„ sin 1″ - - - - - -	4·6856
„ $\operatorname{cosec}\left(\frac{\theta}{2}=6'\right)$ - - - - -	2·7579
„ log (error = 1″) - - - - -	$\bar{1}$·9998
„ $\operatorname{cosec}\left(\frac{\theta}{2}=1^\circ\right)$ - - - - -	1·7579
„ log (error = 0″·1) - - - -	$\bar{2}$·9998
„ $\operatorname{cosec}\left(\frac{\theta}{2}=10^\circ\ 3'\right)$ - - - -	0·7579
„ log (error = 0″·01) - - - -	$\bar{3}$·9998

Hence sextant reading $\theta = 12'$ gives error $= 1''$
" " $= 2^\circ$ " $= 0''\cdot 1$
" " $= 20^\circ\cdot 6'$ " $= 0''\cdot 01$.

The term $(i^2 + h^2)\sin 1'' \tan\frac{\theta}{4}$ can also be rejected when h and i do not exceed $5'$ and $4'$ respectively, in which case this term $= (16+25)60 \times 60'' \sin 1'' \tan\frac{\theta}{4}$. Here the error continually increases with the reading of the sextant; therefore, taking $\theta = 150^\circ$, an unusually large reading, $\frac{\theta}{4} = 37^\circ\ 30'$; and the error $= 41 \times 60 \times 60'' \sin 1'' \tan(37^\circ\ 30') = 0''\cdot 055$.

log 41 - - - - - - -	1·6128
" 360 - - - - - - -	2·5563
" sin 1″ - - - - - -	$\bar{6}$·6856
" tan(37°·30′) - - - - -	9·8850
" 0·055 - - - - - -	$\bar{2}$·7397

Also in this example when $\operatorname{cosec}\frac{\theta}{2} = 41 \tan\frac{\theta}{4}$ these two terms exactly counterbalance each other; the above equation can be reduced at once to $\sin^2\frac{\theta}{4} = \frac{1}{82}$ or $\theta = 25^\circ\ 22'$.

Hence, when the reading of the sextant in this case is $25^\circ\ 22'$ the two errors are equal, and their signs being opposite obliterate each other; readings greater than this give a small positive error which will never exceed $0''\cdot 05$; and those less than 25° will give a small negative error which will be always less than $1''$, so long as the reading of the sextant exceeds $12'$; consequently, by comparing observations made on known angles of different sizes with each other we cannot determine these errors, and it is therefore very important that the adjustments of the index and horizon glasses should be very carefully made, and the errors reduced to the smallest sizes they practically can be. The instrument should be watched during a series of observations to ascertain if any sensible alterations in these adjustments have occurred.

Having determined the value of x in the manner before described for an angle exceeding 120°, unless its value is very large (greater than 1°), and provided h and i do not exceed the limits $i = 4'$ and $h - i = \pm 1'$, and observing that T does not generally differ much from 20°, x may be used as the collimation error of the sextant, without introducing an error of sufficient size to be of any practical importance.

Let θ_1 be an angle less than 120°, and therefore less than θ, the angle used for the determination of x; let c be the collimation error of the sextant x, the value of x corresponding to the reading θ_1 of the sextant, then substituting in equation (F) θ by $\frac{\theta}{2}$, and transposing, we have

$$x = c - \frac{h \sin\left(\frac{\theta}{2} - T\right) + i \sin T}{\sin\frac{\theta}{2}},$$

$$= c - h\cos T + \frac{\sin T}{\sin\frac{\theta}{2}}\left(h\cos\frac{\theta}{2} - i\right)$$

$$= c' + \frac{\sin T}{\sin\frac{\theta}{2}}\left(h\cos\frac{\theta}{2} - i\right).$$

In like manner

$$x_1 = c' + \frac{\sin T}{\sin\frac{\theta_1}{2}}\left(h\cos\frac{\theta_1}{2} - i\right)$$

$$\therefore \quad x - x_1 = -\frac{h\sin T \cdot \sin\frac{\theta-\theta_1}{2}}{\sin\frac{\theta}{2} \cdot \sin\frac{\theta_1}{2}}\left\{1 - \frac{i\cos\frac{\theta+\theta_1}{4}}{h\cos\frac{\theta-\theta_1}{4}}\right\}$$

$$= -\frac{h\sin T \cdot \sin\frac{\theta-\theta_1}{2}}{\sin\frac{\theta}{2} \cdot \sin\frac{\theta_1}{2}} \cdot \sin^2\beta,$$

where $$\cos^2\beta = \frac{i\cos\frac{\theta+\theta_1}{4}}{h\cos\frac{\theta-\theta_1}{4}}.$$

The error caused by taking x for x_1 is $(x^2 - x_1^2)\sin 1'' \tan\frac{\theta_1}{2}$ $= (x + x_1)(x - x_1)\sin 1'' \tan\frac{\theta_1}{2}$.

Let $\theta = 120°$, $\theta_1 = 90°$, $T = 20°$, $h = 5'$, and $i = 4'$;

$$\therefore \quad \frac{\theta+\theta_1}{4} = 52°\,30', \quad \frac{\theta-\theta_1}{2} = 15°, \quad \frac{\theta-\theta_1}{4} = 7°\,30',$$

$$\text{and } \cos^2\beta = \frac{4\cos(52°\,30')}{5\cos(7°\,30')}.$$

$\log\sin\beta$ - -	9·85327	$\log 4$ - - -	0·60206
„ „ - -	9·85327	„ $\cos(52°·30')$ -	9·78445
„ $h''=300''$ - -	2·47712	„ $ac \log 5$ - -	$\bar{1}$·30103
„ $\sin T=20°$ -	9·53405	„ $\sec(7°·30')$ -	0·00373
„ $\sin\frac{\theta-\theta_1}{2}=15°$ -	9·41300		2)19·69127
„ cosec $60°$ - -	0·06247	„ $\log\cos\beta$ - -	9·84563
„ „ $45°$ - -	0·15051		
„ $x_1-x=22''$ -	1·34369		

Let $x=60'=3600''$, $x_1-x=22''$; $\therefore x+x_1=7222''$;
and error $=-22\times7222''\sin 1''\tan 45°=0''·77.$

If $\dfrac{i\cos\frac{\theta+\theta_1}{4}}{h\cos\frac{\theta-\theta_1}{4}}$ is greater than unity, assume it $=\sec^2\beta$, when

$x-x_1$ will be $=\dfrac{h\sin T.\sin\frac{\theta-\theta_1}{4}}{\sin\frac{\theta}{2}.\sin\frac{\theta_1}{2}}\tan^2\beta$, and if we had taken

$\theta_1=60$ we should have found the error amounted to $1''$,
and so, if $\theta_1=40°$, error $=1''$,
also if $\theta_1=30°$, „ $=0''·96$,
and if $\theta_1=20°$, „ $=0''·91$.

From which it is evident that whilst c, i, and h are kept within reasonable limits, the value of x determined from observations made on a large angle, not less than 120°, may be used to correct the collimation error of smaller angles without introducing an error of any practical importance in the result.

Again, referring to equation (8), and taking the terms involving h and i, we find the error depending on h and i

$$=\frac{(i-h)^2}{\sin\frac{\theta}{2}}\sin 1''-(i^2+h^2)\sin 1''\tan\frac{\theta}{4}$$

$$+2c\frac{h\sin\left(\frac{\theta}{2}-T\right)+i\sin T}{\sin\frac{\theta}{2}}\sin 1''\tan\frac{\theta}{2}$$

$$-\left(\frac{h\sin\left(\frac{\theta}{2}-T\right)+i\sin T}{\sin\frac{\theta}{2}}\right)^2\sin 1''\tan\frac{\theta}{2}.$$

In this expression the coefficient of h^2 is

$$\left\{\frac{1}{\sin\frac{\theta}{2}}-\tan\frac{\theta}{4}-\frac{\sin^2\left\{\frac{\theta}{2}-T\right\}}{\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right\}\sin 1''$$

$$=\frac{2\sin 1''}{\sin\theta}\left\{\cos^2\theta-\sin^2\left(\frac{\theta}{2}-T\right)\right\}$$

which vanishes when

$$\cos^2\frac{\theta}{2}=\sin^2\left(\frac{\theta}{2}-T\right)$$

or when $$\frac{\theta}{2}=\frac{\pi}{2}-\frac{\theta}{2}+T$$

" $$\theta=\frac{\pi}{2}+T$$

$$=110^\circ \text{ when } T=20^\circ.$$

In like manner the coefficient of i^2 vanishes when

$$\frac{\theta}{2}=\frac{\pi}{2}-T,$$

or when $$\theta=\pi-2T=140^\circ \text{ when } T=20^\circ.$$

The coefficient of hi is

$$-2\sin 1''\left\{\frac{1}{\sin\frac{\theta}{2}}+\frac{\sin\left(\frac{\theta}{2}-T\right)\sin T}{\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right\}$$

$$=-2\cos T\sin 1''\left\{\frac{\cos T}{\sin\frac{\theta}{2}}+\frac{\sin T}{\cos\frac{\theta}{2}}\right\}$$

which has its smallest value when $\tan^3\frac{\theta}{2}=\cot T$, or when $T=20^\circ$, $\theta=108^\circ\ 57'$.

The term $$2c\frac{h\sin\left(\frac{\theta}{2}-T\right)+i\sin T}{\sin\frac{\theta}{2}}\sin 1''\tan\frac{\theta}{2}$$

$$=2c\left\{h\cos T+\frac{\left(i-h\cos\frac{\theta}{2}\right)\sin T}{\sin\frac{\theta}{2}}\right\}\sin 1''\tan\frac{\theta}{2};$$

the constant part of the coefficient of $\tan\frac{\theta}{2}$ will be included

in the value of x determined as before explained for a large angle of not less than 120°: the remaining part, viz.

$$2c\frac{\left(i-h\cos\frac{\theta}{2}\right)\sin T}{\sin\frac{\theta}{2}}\sin 1''\tan\frac{\theta}{2},$$

need therefore only be considered; this is

$$=2ci\sin T\frac{\sin 1''}{\cos\frac{\theta}{2}}-2ch\sin T\sin 1''.$$

The latter part of this is constant for all values of θ, and will therefore be included in the index error determined in the ordinary way; the other part has its least value when the index bar is at zero, and increases with θ the reading of the sextant. Taking $i=4'$, $h=5'$, $c=30'$, and $T=20°$, it will be found that

when $\theta=$	0°	error	$=1''\cdot43$
$=$	20°	,,	$=1''\cdot45$
$=$	40°	,,	$=1''\cdot53$
$=$	60°	,,	$=1''\cdot65$
$=$	80°	,,	$=1''\cdot87$
$=$	100°	,,	$=2''\cdot23$
$=$	120°	,,	$=2''\cdot86$
$=$	140°	,,	$=4''\cdot19$

The error when $\theta=0$ will be included in the index error, and therefore the uncorrected error for the larger angles will be—

When $\theta=$	20°	uncorrected part	of error	$=0''\cdot02$
$=$	40°	,,	,,	$=0''\cdot10$
$=$	60°	,,	,,	$=0''\cdot22$
$=$	80°	,,	,,	$=0''\cdot44$
$=$	100°	,,	,,	$=0''\cdot79$
$=$	120°	,,	,,	$=1''\cdot43$
$=$	140°	,,	,,	$=2''\cdot36$

Consequently, when the errors of adjustment are within the above limits, they will not sensibly affect sextant angles which do not exceed 120°.

The correction depending on x^2 varies with the tangent of half the reading of the sextant, and continually diminishes with the reading, so that any error made in determining the value of x for an angle exceeding 120° will have a diminished effect on smaller angles calculated from it, and we may conclude generally that angles between 20° and 100° are best adapted for sextant observations.

Having carefully adjusted a sextant so as to make x, h, and i as small as possible, and well within the limits pointed out,

when the value of x has been determined for an angle between 120° and 130°, in the manner before described, we can proceed to determine the values of e and a from the equation,

$$A=\theta-\theta_0+e\left\{\sin\frac{\theta'-a}{2}-\sin\frac{\theta_0'-a}{2}\right\}-x^2\sin 1''\tan\frac{\theta}{2}\ldots\ldots(9)$$

The angle T should be noted on every sextant by its maker; when he has omitted to do so it may be determined by placing the sextant face upwards on a table, so as to see a distant well defined object just over the unsilvered part of the horizon glass when looking through the telescope collar. Bring the reflected image of the same object to coincide with that seen directly, clamp the index bar, screw in the telescope, make the coincidence exact by the tangent screw when necessary, and read off the sextant; place a theodolite so that the image of the distant object reflected from the index glass may be seen over the middle of the horizon glass when looking through its telescope, bring the centre of the cross wires to cut it, and clamp the theodolite; turn the sextant until the cross wires of the telescope of the theodolite can be seen through the telescope of the sextant; unclamp the index bar of the sextant and move it on until the reflected image of the distant object is in one with the cross of the wires of the theodolite telescope as seen through the telescope of the sextant; clamp the index bar of the sextant and read it off. The difference between the two readings of the sextant will be twice the angle T. If a theodolite is not at hand measure the distances between the centres of the index and horizon glasses and the centre of the telescope collar, respectively, which form a triangle; the angle at the centre of the telescope collar, opposite to the side corresponding to the distance between the centres of the index and horizon glasses, will be twice the angle T.

We will now consider the effect produced on the reading of a sextant by the silvered and unsilvered surfaces of the index and horizon glasses not being exactly parallel to each other.

Let the straight lines OS, ON (Fig. 5) be parallel to the normals to the silvered and unsilvered surfaces of the index and horizon glasses of a sextant, respectively; IO parallel to a ray of light incident on the outer surface of the index glass and refracted by it in a direction parallel to rO, then reflected at the silvered surface in a direction parallel to Oq, and emerging from the unsilvered surface in a direction parallel to OC.

Suppose a sphere described about O as a centre, with a radius OS; then, since IO, rO, and NO are in the same plane, IrN is the arc of a great circle; similarly NqC and rSq are arcs of

great circles. Let μ be the index of refraction of the index glass, then by the law of reflection $Sr = Sq$, and by the law of refraction

$$\sin IN = \mu \sin Nr, \text{ and}$$
$$\sin NC = \mu \sin Nq.$$

Now IOC is the observed angle $= IS + SC$ very approximately,

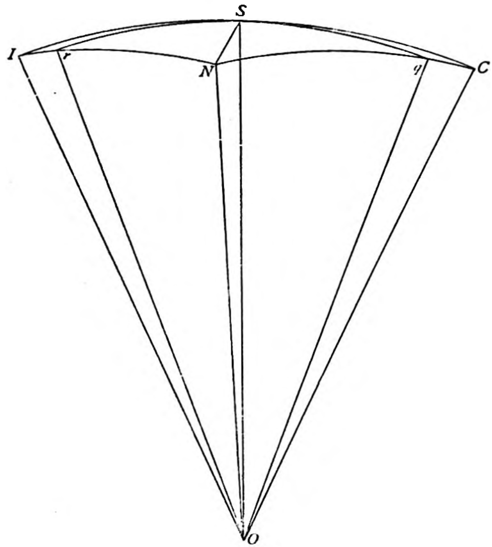


Fig. 5.

since NOS is very small, and the reading of the sextant $= 2SC$. Therefore error $= IS + SC - 2SC = IS - SC$.

Let $SC = \phi$, angle $NSC = q$, and $SN = p$, all expressed in circular measure; and when p is so small with respect to ϕ that its square and higher powers may be neglected,

$$\cos NC = \cos \phi \,.\, \cos p + \sin \phi \,.\, \sin p \cos q$$
$$= \cos \phi + p \sin \phi \,.\, \cos q, \text{ neglecting the}$$

square and higher powers of p;

$$\therefore \qquad NC = \phi - p \cos q \text{ very approximately.}$$

In like manner we may put

$$NI = IS + p \cos q,$$
$$\therefore \qquad NI - NC = IS - \phi + 2p \cos q,$$
$$\therefore \qquad IS - SC = NI - NC - 2p \cos q.$$

Similarly $$Nq = Sq - p\cos q,$$
$$Nr = Sr + p\cos q. \quad \text{But } Sr = Sq;$$

$\therefore$ $$Nq + p\cos q = Nr - p\cos q;$$

$\therefore$ $$\sin Nq + p\cos q\cos Nq = \sin Nr - p\cos q\cos Nr.$$

But $$\sin Nq = \frac{\sin NC}{\mu}$$

and $$\sin Nr = \frac{\sin NI}{\mu};$$

$\therefore$ $$\cos Nq = \frac{\sqrt{\mu^2 - \sin^2 NC}}{\mu}$$

and $$\cos Nr = \frac{\sqrt{\mu^2 - \sin^2 NI}}{\mu};$$

$$\therefore \sin NC + p\cos q\sqrt{\mu^2 - \sin^2 NC} = \sin NI - p\cos q\sqrt{\mu^2 - \sin^2 NI}.$$

In the terms multiplied by p, we may replace NC and NI respectively by ϕ without sensible error, and we shall have

$$NC + p\cos q\frac{\sqrt{\mu^2 - \sin^2\phi}}{\cos\phi} = NI - p\cos q\frac{\sqrt{\mu^2 - \sin^2\phi}}{\cos\phi}$$

$\therefore$ $$NI - NC = 2p\cos q\frac{\sqrt{\mu^2 - \sin^2\phi}}{\cos\phi}.$$

Now error $$IS - SC = NI - NC - 2p\cos q$$
$$= 2p\cos q\left\{\frac{\sqrt{\mu^2 - \sin^2\phi}}{\cos\phi} - 1\right\}.$$

Where θ is the reading of the sextant and T the telescopic angle, $\phi = \frac{\theta}{2} + T$; $2p\cos q$ is constant for the index glass so long as it remains in the same position with respect to the frame which secures it to the sextant; let $\lambda = 2p\cos q$ and the error becomes $= \lambda\left\{\frac{\sqrt{\mu^2 - \sin^2\left(\frac{\theta}{2} + T\right)}}{\cos\left(\frac{\theta}{2} + T\right)} - 1\right\}$; when the reading θ_0, corresponding to the coincidence of the reflected and direct image of a star is taken, the error in this case will be $\lambda\left\{\frac{\sqrt{\mu^2 - \sin^2 T}}{\cos T} - 1\right\}$; and therefore when the index error is applied to the sextant the resulting error

$$= \lambda\left\{\frac{\sqrt{\mu^2 - \sin^2\left(\frac{\theta}{2} + T\right)}}{\cos\left(\frac{\theta}{2} + T\right)} - \frac{\sqrt{\mu^2 - \sin^2 T}}{\cos T}\right\}$$

$= \lambda F(\theta)$, suppose..................................(12);

adding this to equation (9) we have

$$A=\theta-\theta_0+e\left\{\sin\frac{\theta'-a}{2}-\sin\frac{\theta_0'-a}{2}\right\}-x^2\sin 1''\tan\frac{\theta}{2}+\lambda F(\theta) \quad ..(13)$$

To determine λ, the angle between two fixed stars, whose distance is between 120° and 130°, must be very carefully observed, and the index error and collimation errors determined; the index glass must then be taken out of its frame, turned upside down, and replaced in its frame in that position, the sextant carefully adjusted, its index and collimation errors determined, and the angle between the same two stars again observed with great care. By reversing the index glass the sign of λ is changed, and the difference between the two observations properly corrected will $=2\lambda F(\theta)$, and $F(\theta)$ being known, that of λ will immediately follow.

When measuring the distance between two stars for the purpose of determining the error of a sextant, it is important to consider the effect refraction has on the apparent position of the stars, and so time the observations that the difference between the true and apparent distances of the stars from this cause may have its smallest possible size, and that the altitudes of the stars may be sufficiently large to ensure the tables of refraction being trustworthy.

Let Z (Fig. 6) be the zenith of the place, P the pole of the

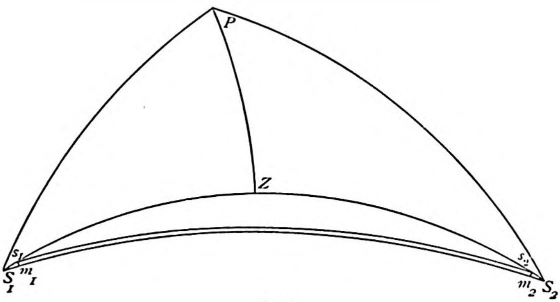


FIG. 6.

heavens, S_1 and S_2 the true places of two stars, s_1 and s_2 their apparent places due to refraction.

Let also PS_1 the polar distance of $S_1=\Delta_1$,
" PS_2 " " $S_2=\Delta_2$,
S_1S_2 the true distance of the stars $=D$.
ZS_1, the zenith distance of $S_1=Z_1$,
ZS_2, " " $S_2=Z_2$.

It has been found from careful observations, combined with theoretical considerations, that the effect of refraction on a star S_1 is to elevate it in the vertical circle S_1Z by a small distance S_1s_1, which, so long as Z_1 does not exceed 70°, will be accurately expressed by $r \tan z_1 +$ terms in its higher positive powers, where $r \tan z_1$ is so small that its square and higher powers may be neglected as being of no practical importance; and that r may be expressed as follows,

$$r = \frac{b}{29{\cdot}6} \times \frac{1{\cdot}0375 \times 57''{\cdot}82}{1 + 0{\cdot}002083(t-32)},$$

where b is the reading of the barometer in inches, and t the reading of a Fahrenheit thermometer in degrees.

On these conditions we have

$$S_1s_1 = r \tan z_1, \text{ and } S_2s_2 = r \tan z_2.$$

Let s_1m_1 and s_2m_2 be arcs of great circles perpendicular to S_1S_2, and passing through s_1 and s_2 respectively.

Hence $s_1s_2 = m_1m_2$ very approximately.

In the right-angled triangle $S_1m_1s_1$

$$\cos S_1 = \cot S_1s_1 \tan S_1m_1;$$

$\therefore$ $$S_1m_1 = S_1s_1 \cos S_1 = r \tan z_1 \cos S_1;$$

similarly $$S_2m_2 = r \tan z_2 \cos S_2;$$

$\therefore$ apparent distance $s_1s_2 = m_1m_2 = D - S_1m_1 - S_2m_2$

$$= D - r(\tan z_1 \cos S_1 + \tan z_2 \cos S_2) \ldots (14).$$

The stars are here supposed to be on opposite sides of the meridian, which is necessary to suit our present purpose.

In the triangle S_1ZS_2 we have

$$\cos z_2 = \cos z_1 \cos D + \sin z_1 \sin D \cos S_1;$$

$\therefore$ $$\tan z_1 \cos S_1 = \frac{1}{\sin D} \cdot \frac{\cos z_2}{\cos z_1} - \cot D.$$

In like manner, $\tan z_2 \cos S_2 = \dfrac{1}{\sin D} \cdot \dfrac{\cos z_1}{\cos z_2} - \cot D.$

Substituting these values in (14),

$$\text{apparent distance} = D + 2r \cot D - \frac{r}{\sin D}\left\{\frac{\cos z_1}{\cos z_2} + \frac{\cos z_2}{\cos z_1}\right\}$$

$$= D + 2r\{\cot D - \operatorname{cosec} D \operatorname{cosec} 2Z\} \ldots\ldots\ldots (15)$$

where $\tan Z = \dfrac{\cos z_1}{\cos z_2}$.

Now $\operatorname{cosec} D \operatorname{cosec} 2Z - \cot D$ has its smallest value when $2Z = 90°$ or when $Z = 45°$, in which case $z_1 = z_2$ and

$$\text{apparent distance} = D - 2r \tan\frac{D}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Consequently, the best time for observing the distance between two stars, so that refraction may have its least possible

effect, and alter slowest with the time, is when the altitudes of the two stars are equal.

To determine the sidereal time when $z_1 = z_2$.

Let A_1 and A_2 be the right ascensions of the two stars, h_1 and h_2 their hour angles on opposite sides of the meridian, and let $P = A_2 - A_1 = h_1 + h_2$ and $PZ = c$. Then in order that z_1 may be equal to z_2, we must have

$$\cos c \cos \Delta_1 + \sin c \sin \Delta_1 \cos h_1 = \cos c \cos \Delta_2 + \sin c \sin \Delta_2 \cos h_2;$$

$$\therefore \quad \sin \Delta_1 \cos h_1 - \sin \Delta_2 \cos (P - h_1) = \cot c \{\cos \Delta_2 - \cos \Delta_1\};$$

$$\therefore \quad \{\sin \Delta_1 - \sin \Delta_2 \cos P\} \cos h_1 - \sin \Delta_2 \sin P \sin h_1$$

$$= 2 \cot c \sin \frac{\Delta_1 - \Delta_2}{2} \sin \frac{\Delta_1 + \Delta_2}{2},$$

and putting $\tan y = \dfrac{\sin \Delta_1 - \sin \Delta_2 \cos P}{\sin \Delta_2 \sin P}$ we find

$$\sin (y - h_1) = 2 \cot c \sin \frac{\Delta_1 - \Delta_2}{2} . \sin \frac{\Delta_1 + \Delta_2}{2} \ldots\ldots\ldots\ldots(17)$$

To calculate y put $\qquad \cot k = \dfrac{\sin \Delta_1}{\sin \Delta_2 \sin P} \ldots\ldots\ldots\ldots\ldots\ldots(18)$

and we have $\qquad \tan y = \dfrac{\sin (P - k)}{\sin k \sin P} \ldots\ldots\ldots\ldots\ldots\ldots(19)$

in a form adapted to logarithms. Having determined h_1, we have the sidereal time $A_1 + h_1$ at which the observation should be taken.

Suppose D_1 and D_2 be the apparent distances of two stars corresponding to values Z_1 and Z_2 of Z, respectively, and to values r_1 and r_2 of r, putting δr for $r_1 - r_2$, substituting these values in equation (14), and taking the difference of the two results, we have

$$D_1 - D_2 = \frac{2r_1}{\sin D}\{\operatorname{cosec} 2Z_2 - \operatorname{cosec} 2Z_1\} - \frac{2\delta r}{\sin D}\{\operatorname{cosec} 2Z_2 - \cos D\}$$

$$= \frac{4r_1 \sin (Z_1 - Z_2) \cos (Z_1 + Z_2)}{\sin D \sin 2Z_1 . \sin 2Z_2} - \frac{2\delta r \sin^2 n}{\sin D \sin 2Z_2} \ldots\ldots\ldots(20)$$

where $\cos^2 n = \sin 2Z_2 \cos D$.

This is useful when combining or comparing with each other observations made at different sidereal times, and in different states of the atmosphere.

For our present purpose we use two values of D, one of which should not differ much from 60°, and the other ought to be intermediate in value between 120° and 130°; consequently the following expression will suit for calculating the value of D:

$$\cos D = \cos \Delta_1 \cos \Delta_2 + \sin \Delta_1 \sin \Delta_2 \cos P = \frac{\cos \Delta_1 \cos (\Delta_2 - x)}{\cos x} \ldots(21)$$

where $\tan x = \tan \Delta_1 \cos P$.

To determine Z we have

$$\tan Z = \frac{\cos z_1}{\cos z_2} = \frac{\cos \Delta_1 + \sin \Delta_1 \tan c \cos h_1}{\cos \Delta_2 + \sin \Delta_2 \tan c \cos h_2}$$

$$= \frac{\cos (\Delta_1 - m_1) \cos m_2}{\cos (\Delta_2 - m_2) \cos m_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

where $\tan m_1 = \tan c \cos h_1$, and $\tan m_2 = \tan c \cos h_2$.

We have also, if we wish to determine z_1,

$$\cos z_1 = \cos (\Delta_1 - m_1) \sec m_1 \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

and a similar expression for finding z_2.

We will now take the following example to show the practical application of the foregoing:

On the 13th March, 1864, in latitude 36° 41′ N., the apparent distance of α Geminorum from α Ophiuchi was observed; six good contacts were made on wire (1) of the telescope of a sextant, of which the telescopic angle was 20° 10′, the mean of which gave

Sidereal time $12^h\ 40^m\ 15^s$, distance 127° 4′ 10″·2.

Six good contacts were then carefully made on wire (2) of the same sextant, the mean of which gave

Sidereal time $12^h\ 49^m\ 26^s$, distance 127° 3′ 46″·3.

Six contacts were then made on wire (1), which gave

Sidereal time $13^h\ 0^m\ 43^s$, distance 127° 4′ 9″·6.

Lastly, six contacts were again made on wire (2), which gave

Sidereal time $13^h\ 11^m\ 28^s$, distance 127° 3′ 45″·5.

The observed distance of the wires (1) and (2), given by the mean of twelve good observations, was 4725″.

On the 14th March, 1864, at the same place, with the same sextant, the following observations were made:

1. The distance between α Geminorum and β Leonis, twelve good contacts, the mean of which gave

Sidereal time $9^h\ 55^m\ 47^s{\cdot}3$, distance 60° 41′ 27″·3.

2. Twelve good coincidences of the reflected and direct images of a bright star gave a mean reading of 54″·23.

The distance between α Geminorum and α Ophiuchi was then observed, twelve good contacts being taken, the mean of which gave

Sidereal time $12^h\ 53^m\ 47^s{\cdot}3$, distance 127° 3′ 24″.

Next, twelve coincidences of the direct and reflected images of a bright star were taken, the mean of the readings of the sextant being 54″·77.

The distance between η Ursae Majoris and α Ophiuchi was also observed, twelve contacts being taken, the mean of which gave

Sidereal time $16^h\ 10^m\ 22^s{\cdot}5$, distance 59° 7′ 46″.

The readings of the barometer and thermometer were as follows:

At 10^h sidereal time,	barometer	30·02 in.,	Fahr. therm.	60·0°
13 „	„	30·045 „	„	59·25
16 „	„	30·06 „	„	58·0

On the 15th March the index glass was taken out, turned upside down, and replaced in that position and carefully adjusted; during the night the following observations were made for collimation error. The distance between α Geminorum and α Ophiuchi was observed on wire (1), the mean of six good contacts gave

Sidereal time $12^h\ 40^m\ 10^s{\cdot}4$, distance $127°\ 3'\ 58''{\cdot}6$,

a similar set of observations on wire (2) gave

Sidereal time $12^h\ 49^m\ 30^s{\cdot}2$, distance $127°\ 3'\ 37''{\cdot}4$.

Six contacts on wire (1) were then taken, the mean of which gave

Sidereal time $13^h\ 0^m\ 47^s{\cdot}2$, distance $127°\ 3'\ 59''{\cdot}2$.

Then six more on wire (2) gave

Sidereal time $13^h\ 11^m\ 34^s$, distance $127°\ 3'\ 36''{\cdot}8$.

Observed distance between the wires, mean of twelve observations, gave $4724''{\cdot}2$. It is not absolutely necessary to take the time when observing for collimation error; but I prefer doing so because it points out the situation of the stars.

On the 16th March the following observations were made with the same sextant by the same observer.

Distance between α Geminorum and β Leonis, the mean of twelve good contacts gave

Sidereal time $9^h\ 56^m\ 2^s$, distance $60°\ 41'\ 25''{\cdot}7$.

Twelve good coincidences of the reflected with the direct image of a bright star were then taken, the mean of which gave $52''{\cdot}47$.

The distance between α Geminorum and α Ophiuchi was then observed, mean of twelve good contacts gave

Sidereal time $12^h\ 54^m\ 34^s$, distance $127°\ 3'\ 19''$.

Then twelve coincidences for index error were taken, the mean of which gave $52''{\cdot}1$.

The distance between η Ursae Majoris and α Ophiuchi was next observed, twelve good contacts being read, the mean of which gave

Sidereal time $16^h\ 10^m\ 54^s$, distance $59°\ 7'\ 42''{\cdot}1$.

At 10^h sidereal time,	barometer	30·06 in.,	Fahr. therm.	60°
13 „	„	30·10 „	„	60
16 „	„	30·09 „	„	58·5

First determine the collimation error, or the value of x, from the observations made on 13th March. We have

mean of distances observed on No. 1 wire,		$127°\ 4'\ 9''{\cdot}9$
„ „	No. 2 „	$127\ \ 3\ \ 45\ {\cdot}9$
Difference - - - - -		$24''$

Hence $$x=\frac{24''}{9450 \sin 1'' \tan(63°\ 32')}=260''{\cdot}8.$$

After the index glass was reversed the observations made on 15th March gave

$$x=\frac{21''{\cdot}8}{9448{\cdot}8 \sin 1'' \tan(63°\ 32')}=237''.$$

On the 14th March, 1864, the right ascensions of the stars used in the foregoing observations were

α Geminorum,	R.A.	7^h	25^m	$57^s{\cdot}5$,	declination	$32°$	$10'$	$53''{\cdot}3$	N.
β Leonis, - -	„	11	42	9 ·9,	„	15	19	38	„
η Ursae Majoris,	„	13	42	13 ·1,	„	49	59	15 ·8	„
α Ophiuchi, -	„	17	28	38 ·6,	„	12	39	33 ·8	„

Before proceeding on shore to make the observations, the time at which the altitudes of α Geminorum and β Leonis was determined as follows:

Referring to equations (17), (18), and (19), observing that $P=4^h\ 16^m\ 12^s{\cdot}4$; $\Delta_1=57°\ 49'\ 6''{\cdot}7$; $\Delta_2=74°\ 40'\ 22''$,

from equation (18) $$\cot k=\frac{\sin(57°\ 49'\ 7'')}{\sin(74°\ 40'\ 22'')\sin(4^h\ 16^m\ 12^s{\cdot}4)}.$$

log cosec $(74°\ 40'\ 22'')$, - - -	0·015728
„ $(4^h\ 16^m\ 12^s{\cdot}4)$, - - -	0·046149
log A, - - - - - -	0·061877
„ sin $(57°\ 19'\ 7'')$, - - -	9·927558
„ cot $k(=45°\ 41'\ 48'')$, - -	9·989435

$$P=64°\ \ 3'\ \ 6''$$
$$P-k=18\ \ 21\ \ 18$$

By equation (19) $$\tan y=\frac{\sin(18°\ 21'\ 18'')}{\sin(45°\ 41'\ 48'')\sin(64°\ 3'\ 6'')}$$

log sin $(18°\ 21'\ 18'')$, - - -	9·498178
log cosec $(45°\ 41'\ 48'')$, - - -	0·145297
„ $(64°\ 3'\ 6'')$, - - -	0·046149
log tan y $(=26°\ 4'\ 30'')$, - - -	9·689624

By equation (17)

$$\sin(y-h_1)=2\cot c\,.\,\cos y\,\frac{\sin\frac{\Delta_1-\Delta_2}{2}\,.\,\sin\frac{\Delta_1+\Delta_2}{2}}{\sin\Delta_2\sin P}.$$

log 2, - - - - - -	0·301030
„ cot (53° 19′), - - - -	9·872112
„ cos (26° 4′ 30″), - - -	9·953382
„ sin (8° 25′ 37″), - - -	9·165987
„ sin (66° 14′ 44″), - - -	9·942846
„ A, - - - - - -	0·061877
log sin (h_1-y), - - - -	9·297234

Δ_2 being larger than Δ_1 we have changed their signs and also those of h_1 and y on the other side of the equation, hence

$$h_1-y=11°\ 26'\ \ 7''$$
$$y=26\ \ \ 4\ \ 30$$

Therefore $$h_1=37°\ 30'\ 37''$$

Hence, when the hour angle of α Geminorum after it has passed the meridian is $2^h\ 30^m\ 2^s{\cdot}5$, its altitude is equal to that of β Leonis on the east side of the meridian.

Consequently,

Right ascension of α Geminorum, -	$7^h\ 25^m\ 57^s{\cdot}5$
Hour angle west of meridian, - -	2 30 2 ·5
Sidereal time, - - -	9 56 0

In a similar manner the sidereal time at which α Geminorum and α Ophiuchi have the same altitude was found to be $13^h\ 9^m\ 30^s{\cdot}8$, and that at which η Ursae Majoris and α Ophiuchi have the same altitude was $16^h\ 10^m\ 44^s{\cdot}4$.

To find the true distances of the stars equation (21) was used, which gave

True distance, α Geminorum and β Leonis	= 60° 41′ 24″·4
„ „ α Ophiuchi	=127 5 59
„ η Ursae Majoris and „	= 59 7 40

It is convenient to tabulate the values of r for different states of the atmosphere, derived from the formula

$$r=\frac{b}{29{\cdot}6}\times\frac{1{\cdot}0375\times 57''{\cdot}82}{1+0{\cdot}002083(t-32)}$$

as follows:

TABLE GIVING THE VALUES OF r.

Barometer, Inches.	Thermometer, Fahrenheit Degrees.					
	32°	42°	52°	62°	72°	82°
29·6	$r=59''·99$	58″·76	57″·59	56″·46	55″·37	54″·33
·7	60 ·19	58 ·96	57 ·78	56 ·65	55 ·56	54 ·51
·8	60 ·39	59 ·16	57 ·98	56 ·84	55 ·75	54 ·70
·9	60 ·60	59 ·36	58 ·17	57 ·03	55 ·93	54 ·88
30·0	60 ·80	59 ·56	58 ·37	57 ·22	56 ·12	55 ·06
·1	61 ·00	59 ·76	58 ·56	57 ·41	56 ·31	55 ·25
·2	61 ·20	59 ·96	58 ·76	57 ·61	56 ·50	55 ·43

Before determining the value of λ from the observations made on the 14th and 16th of March, it will be convenient to refer to equation (13).

Suppose therefore that D_1 be the apparent distance between two stars when r_1 was the value of r, θ_1, ${}^1\theta_0'$, θ_1', and ${}^1\theta_0'$, and x_1 the values of θ, θ_0, etc., corresponding thereto; D_2 the apparent distance of the same stars at another time when θ_2, ${}^2\theta_0$, θ_2', ${}^2\theta_0'$, x_2 and r_2 were the respective values of θ, θ_0, θ_1', θ_0', x and r. Then substituting in (13) we shall have

$$D_1=\theta_1-{}^1\theta_0+e\left\{\sin\frac{\theta_1'-a}{2}-\sin\frac{{}^1\theta_0'-a}{2}\right\}-x_1^2\sin 1''\tan\frac{\theta_1}{2}+\lambda F(\theta_1).$$

Supposing the index glass reversed in the interval and the sextant readjusted, we shall have

$$D_2=\theta_2-{}^2\theta_0+e\left\{\sin\frac{\theta_2'-a}{2}-\sin\frac{{}^2\theta_0'-a}{2}\right\}-x_2^2\sin 1''\tan\frac{\theta_2}{2}-\lambda F(\theta_2).$$

Since D_1 and D_2 do not differ much from each other $\therefore\ \dfrac{\theta_1+\theta_2}{2}$ $=\theta$ will differ so little from either θ_1 or θ_2 that it may be substituted for them respectively in the terms multiplied by the small quantities $x^2\sin 1''$, and λ, and observing that θ_1' and θ_2' do not differ much from each other, or from θ' their arithmetic mean, and that ${}^1\theta_0'$ and ${}^2\theta_0'$ do not differ much from θ_0' their arithmetic mean, we may substitute these values for θ_1', ${}^1\theta_0'$, θ_2', and ${}^2\theta_0'$ in the terms multiplied by the small quantity e. When this is done we shall have

$$D_1=\theta_1-{}^1\theta_0+e\left\{\sin\frac{\theta'-a}{2}-\sin\frac{\theta_0'-a}{2}\right\}-x_1^2\sin 1''\tan\frac{\theta}{2}+\lambda F(\theta),$$

$$D_2=\theta_2-{}^2\theta_0+e\left\{\sin\frac{\theta'-a}{2}-\sin\frac{\theta_0'-a}{2}\right\}-x_2^2\sin 1''\tan\frac{\theta}{2}-\lambda F(\theta);$$

taking the difference we have

$$D_1-D_2=\theta_1-\theta_2-{}^1\theta_0+{}^2\theta_0-(x_1+x_2)(x_1-x_2)\sin 1''\tan\frac{\theta}{2}+2\lambda F(\theta)\,;$$

comparing this with equation (20) we have

$$\frac{4r_1\sin(Z_1-Z_2)\cos(Z_1+Z_2)}{\sin D\ \sin 2Z_1\,.\,\sin 2Z_2}-\frac{2\delta r\sin^2 n}{\sin D\,.\,\sin 2Z_2}$$

$$=\theta_1-\theta_2-{}^1\theta_0+{}^2\theta_0-(x_1+x_2)(x_1-x_2)\sin 1''\tan\frac{\theta}{2}+2\lambda F(\theta)$$

$$\therefore\quad \lambda=\frac{\dfrac{4r_1\sin(Z_1-Z_2)\cos(Z_1+Z_2)}{\sin D\,.\,\sin 2Z_1\,.\,\sin 2Z_2}-\dfrac{2\delta r\sin^2 n}{\sin D\,.\sin 2Z_2}}{2F(\theta)}$$

$$\frac{-\theta_1+\theta_2-{}^1\theta_0+{}^2\theta_0-(x_1+x_2)(x_1-x_2)\sin 1''\tan\dfrac{\theta}{2}}{2F(\theta)}\ \ldots\ldots\ \ldots\ldots\ (24).$$

We will now proceed to calculate the value of λ from the foregoing observations: first, the observed distance between α Geminorum and β Leonis.

$\theta_1=60°\ 41'\ 17''\cdot3$, sidereal time $9^h\ 55^m\ 47^s\cdot3$.

${}^1\theta_0=54''\cdot5$, $x_1=260''\cdot8$, r_1 for bar. $30^{in}\cdot02$ and therm. $60°=57''\cdot49$.

$\theta_2=60°\ 41'\ 15''\cdot7$, at sidereal time $9^h\ 56^m\ 2^s$.

${}^2\theta_0=52''\cdot3$, $x_2=237''$, bar. $30^{in}\cdot06$ and therm. $61°$, gave $r_2=57''\cdot44$.

The sidereal time, taken to the nearest second, gave for the calculation of Z_1 $h_1=2^h\ 29^m\ 50^s$, and $h_2=1^h\ 46^m\ 23^s$.

For the calculation of Z_2 we have $h_2=1^h\ 46^m\ 8^s$, and $h_1=2^h\ 30^m\ 5^s$; $\Delta_1=57°\ 49'\cdot7$, $\Delta_2=74°\ 40'\ 22''$, and $c=53°\ 19'$.

Calculation to 4 places of logarithms, and to the nearest 15″, which is sufficiently near for our purpose; using equation (22)

For Z_1.		For Z_2.	
log cos ($2^h\ 29^m\ 50^s$) -	9·8997	log cos ($2^h\ 30^m\ 5^s$) -	9·8993
„ tan (53° 19′) -	0·1279	.... -	0·1279
„ tan m_1 (46° 49′ 30″)	0·0276	„ tan m_1 (46° 47′ 45″)	0·0272
„ log cos (1° 46′ 23″)	9·9514	„ cos (1° 46′ 8″) -	9·9517
„ tan (53′ 19″) -	0·1279	.... -	0·1279
„ tan m_2 (50° 12′ 15″)	0·0793	„ tan m_2 (50° 13′ 30″)	0·0796

$\Delta_1=59°\ 49'\ \ 0''$	$\Delta_1=57°\ 49'\ \ 0''$
$m_1=46°\ 49'\ 30''$	$m_1=46°\ 47'\ 45''$
$\Delta_1-m_1=10°\ 59'\ 30''$	$\Delta_1-m_1=11°\ \ 1'\ 15''$

$\Delta_2 = 74°\ 40'\ 15''$	$\Delta_2 = 74°\ 40'\ 15''$
$m_2 = 50°\ 12'\ 15''$	$m_2 = 50°\ 13'\ 30''$
$\Delta_2 - m_2 = 24°\ 28'\ 0''$	$\Delta_2 - m_2 = 24°\ 26'\ 45''$

log cos m_2 - -	9·8062	- - - - 9·8060
„ $(\Delta_1 - m_1)$ - -	9·9920	- - - - 9·9919
„ sec m_1 - -	0·1648	- - - - 0·1645
„ „ $(\Delta_2 - m_2)$ -	0·0409	- - - - 0·0408
log tan $Z_1 (= 45°\ 15'\ 30'')$	0·0039	log tan $Z_2 (= 45°\ 13'\ 0'')$ 0·0032

$$\therefore \qquad 2Z_1 = 90°\ 31',\ \ Z_1 - Z_2 = 2'\ 30'',\ \ 2Z_2 = 90°\ 26',$$
$$(Z_1 + Z_2) = 90°\ 28'\ 30''.$$

Substitute these values in the terms of equation (24), and calculate them as follows:

First. Taking the term

$$4r_1 \sin(Z_1 - Z_2)\cos(Z_1 + Z_2)\operatorname{cosec} D \,.\, \operatorname{cosec} 2Z_1 \,.\, \operatorname{cosec} 2Z_2.$$

$4r_1 = 229·96$, log	- - - - - -	2·3617
	„ sin (2′ 30″) - - - -	6·8617
(negative)	„ cos (90° 28′ 30″) - - -	7·9186
„	„ cosec (60° 41′ 30″) - - -	0·0595
„	„ (90° 31′ 0″) - - - -	0·0000
„	„ (90° 26′ 0″) - - - -	0·0000
„	„ −0″·0026 - - - -	$\bar{3}$·2015

The cosine of (90° 28′ 30″) being negative, whilst the other terms are positive, gives a negative sign to the product. This term, as might have been supposed from Z_1 and Z_2 being nearly equal, is too small to have any practical effect on the value of λ; but we have it kept in sight here to show the process, when, from uncontrollable circumstances, the observations have been taken at sidereal times which differ considerably from each other, and in consequence this term becomes important.

Secondly. The term $2\delta r \sin^2 n \,.\, \operatorname{cosec} D \,.\, \operatorname{cosec} Z_2$, where

$$\cos^2 n = \sin 2Z_2 \cos D.$$

Here $\delta r = r_1 - r_2 = 0''·05$; $\therefore\ 2\delta r = 0''·1$.

log cos (60° 41′ 30″) -	9·6898	log sin n - - -	9·8539
„ sin (90° 26′) -	0·0000	- - - -	9·8539
		„ cosec (60° 41′ 30″)	0·0595
2 log cos n - -	19·6898	„ „ (90° 26′) -	0·0000
log cos n - -	9·8449	„ 0·1 - - -	$\bar{1}$·0000
		log 0·0585 - -	$\bar{2}$·7673

Hence this term is $=0''\cdot0585$, and the same, or rather a similar remark, applies to this term, that though of little or no importance in this case, the state of the atmosphere may sometimes change so greatly as to make $r_1 - r_2$, and therefore this term, sufficiently large to have an important effect on the value of λ.

Thirdly.

$$\theta_1 - \theta_2 - {}^1\theta_0 + {}^2\theta_0 = 60^\circ\ 41'\ 27''\cdot3 - 60^\circ\ 41'\ 25''\cdot7 - 54''\cdot5 + 52''\cdot3$$
$$= -0''\cdot6$$

Fourthly. The term $(x_1+x_2)(x_1-x_2)\sin 1''\tan\frac{\theta}{2}$:

$x_1+x_2 = 497''\cdot8$, log	- - - -	2·6970
$x_1-x_2 = \ 23\ \cdot8$ „	- - - -	1·3766
„ sin 1″	- - -	4·6856
„ tan (30° 20′ 45″)	-	9·7675
log (0·034)	- - - - -	$\bar{2}$·5267

$\therefore$ this term $= -0''\cdot034$.

Lastly. $$F(\theta) = \frac{\sqrt{(1\cdot5)^2 - \sin^2(50^\circ\ 31')}}{\cos(50^\circ\ 31')} - \frac{\sqrt{(1\cdot5)^2 - \sin^2(20^\circ\ 10')}}{\cos(20^\circ\ 10')}.$$

Let $$\cos u = \frac{\sin(50^\circ\ 31')}{1\cdot5} \text{ and } \cos u' = \frac{\sin(20^\circ\ 10')}{1\cdot5},$$

then $$F(\theta) = 1\cdot5\left\{\frac{\sin u}{\cos(50^\circ\ 31')} - \frac{\sin u'}{\cos(20^\circ\ 10')}\right\}.$$

log sin (50° 31′) - -	9·8875	log sec (50° 31′) -	0·1966
„ 1·5 - - -	0·1761	- - - -	0·1761
log cos u - - -	9·7114	log sin u - -	9·9332
		log 2·023 - -	0·3059
log sin (20° 10′) - -	9·5375	log sec (20° 10′) -	0·0275
„ 1·5 - - -	0·1761	- - - -	0·1761
log cos u' - - -	9·3614	log sin u' - -	9·9882
		log 1·555 - -	0·1918

$$\therefore \qquad F(\theta) = 2\cdot023 - 1\cdot555 = 0\cdot468.$$

Substituting this value in (2) we find

$$\lambda = \frac{-0''\cdot0026 - 0''\cdot0585 + 0''\cdot6 + 0''\cdot034}{0''\cdot936} = \frac{0''\cdot573}{0''\cdot936}.$$

In a similar manner the observations on α Geminorum and

α Ophiuchi give $\lambda = -\frac{2''\cdot00}{17''\cdot34}$; and those on η Ursae Majoris and α Ophiuchi give $\lambda = -\frac{1''\cdot77}{0''\cdot88}$: meaning these according to value, we have from the first

$$(0\cdot936)^2\lambda = \quad 0''\cdot57 \times \ 0\cdot936$$
$$(17\cdot34)^2\lambda = -2\ \cdot10 \times 17\cdot34$$
$$(0\cdot88)^2\lambda = -1\ \cdot77 \times \ 0\cdot88,$$

or

$$0\cdot876 \times \lambda = \quad 0''\cdot534$$
$$300\cdot62 \ \times \lambda = -36\ \cdot41$$
$$0\cdot774 \times \lambda = -\ 1\ \cdot558$$

$$302\cdot27 \ \times \lambda = -37''\cdot434$$

$$\therefore \qquad \lambda = -\frac{37434}{302270} = -0''\cdot124.$$

We can now determine the values of e and a from the same observations, on the supposition that they have not been altered by the process of taking out and inverting the index glass.

The true distance of α Geminorum and β Leonis was, as previously stated, 60° 41′ 24″·4; to this the correction for refraction given by equation (14) must be applied; we therefore put

$$D_1 = 60°\ 41'\ 24''\cdot4 - \frac{114''\cdot98}{\sin(61°\ 42')\ \sin(90°\ 31')} \times \{1 - \sin(90°\ 31') . \cos(61°\ 42')\}.$$

Observing that sin (90° 31′) does not differ sensibly from unity, this latter term may be reduced without introducing a sensible error to 114″·98 × tan (30° 51′).

∴ log 114″·98	- - - - -	2·0602
„ tan (30°·51)	- - - -	9·7675
log 67·26	- - - - - -	1·8277

$$\therefore \qquad 14 \text{ March}, D_1 = 60°\ 41'\ 24''\cdot4 - (1'\ 7''\cdot36) = 60°\ 40'\ 17''\cdot14.$$
$$\text{Similarly}, 16\ „ \quad D_1 = 60\ \ 41\ \ 24\ \cdot4 - (1\ \ 7\ \cdot2) \ = 60\ \ 40\ \ 17\ \cdot2,$$
$$\text{Mean} \qquad D_1 = 60\ \ 40\ \ 17\ \cdot2.$$

14 obs. gave observed dist.

$$60°\ 41'\ 27''\cdot3 - 54''\cdot5 - 0''\cdot19 - 0''\cdot06 = 60°\ 40'\ 32''\cdot6,$$

16 obs. gave observed dist.

$$60°\ 41'\ 25''\cdot7 - 52''\cdot3 - 0''\cdot16 + 0''\cdot06 = 60°\ 40'\ 33''\cdot3,$$
$$\text{mean} = 60°\ 40'\ 33''.$$

Referring to equations A and B we have

$$2d_1 = 60°\ 40'\ 17''\cdot2 - 60°\ 40'\ 33'' = -15''\cdot8;$$
$$\therefore \qquad d_1 = -7''\cdot9.$$

α Geminorum and α Ophiuchi by observations made on

14 March, $Ap.D.=127°\,3'\,24'' \;-54''\!\cdot\!5-0''\!\cdot\!7-1''\!\cdot\!1=127°\,2'\,27''\!\cdot\!7$
16 „ $=127\;\;3\;\;19\cdot5-52\cdot3-0\cdot5+1\cdot1=127\;\;2\;\;27\cdot8$

Calculated ap. distance, 14 March, -	127° 2′ 7″·4
„ „ 16 „ -	127 2 7 ·3
Mean of observed apparent distance, -	127 2 27 ·75
„ calculated „ -	127 2 7 ·35

$\therefore \qquad 2d_2 = \qquad -20''\!\cdot\!4$

and $\qquad d_2=-10''\!\cdot\!2.$

Therefore taking the mean of the observations made on the 14th March and on the 16th, we have

$$\theta_1=\;60°\,41'\,26'',\;\theta_0=53''\!\cdot\!4,\;\theta_1'=\;62°\;\;6',\;\theta_0'=53'$$
$$\theta_2=127\;\;\;3\;\;22\;,\;\theta_0=53\;\cdot4,\;\theta_2'=130\;25,\;\theta_0'=53'$$

$$\therefore \qquad \phi_1=\frac{62°\,6'-53''}{4}=15°\,18'':\;\psi_1=\frac{62°\,6'+53''}{4}=15°\,45':$$

$$\phi_2=\frac{130°\,25'-53''}{4}=32°\,23'':\;\psi_2=\frac{130°\,25'+53''}{4}=32°\,50'.$$

Hence $\beta=\dfrac{7\cdot9\times\sin(32°\,23')}{10\cdot2\times\sin 15°\,18'}$.

Log 7·9, - - -	0·8976
„ sin (32° 23′), -	9·7288
„ cosec (15° 18′), -	0·5786
AC's of log 10·2, -	8·9914
log β, - - -	0·1964

log β, - - -	0·1964	log β, - - -	0·1964
„ cos (32° 50′), -	9·9244	„ sin 32° 50′, - -	9·7342
„ sec (15° 45′), -	0·0166	„ cosec 15° 45′, -	0·5663
2 log sec x, - -	0·1374	2 log sec y, - -	10·4969
∴ log sec x, - -	10·0687	∴ log sec y, - -	10·2484

log cot 15° 45′, -	0·5497	$\therefore \qquad \frac{\alpha}{2}=148°\,20'$
„ tan x, - -	9·7854	
„ tan x, - -	9·7854	$\therefore \qquad \psi_2-\frac{\alpha}{2}=32°\,50'-151°\,20'$
„ cot y, - -	9·8548	$=-115°\,30'$
„ cot y, - -	9·8348	
„ tan $\frac{\alpha}{2}$ (negative),	9·790	$\therefore \qquad e=10''\!\cdot\!2\,\mathrm{cosec}\,(32°\,23')\times\sec(64°\,30')$

log 10·2, corrected, - - - -	1·0086
„ cosec (32° 23′), - - - -	0·2712
„ sec (64° 30′), - - - -	0·3660
„ $e(=44''\cdot24)$, - - - -	1·6458

Next, taking η Ursae Majoris and α Ophiuchi.

14 March, observed distance corrected

$$=59^\circ\ 7'\ 46''-54''\cdot5-0''\cdot25-0''\cdot05=59^\circ\ 6'\ 51''\cdot2;$$

16 March, observed distance corrected

$$=59^\circ\ 7'\ 42''-52''\cdot3-0''\cdot25+0''\cdot05=59^\circ\ 6'\ 49''\cdot5.$$

14 March, calculated apparent distance $=59^\circ\ 6'\ 34''\cdot4$

16 „ „ „ $=59\ \ 6\ \ 34\ \cdot4$

Mean of corrected observed distances $=59^\circ\ 6'\ 50''\cdot35$

„ calculated apparent distances $=59\ \ 6\ \ 34\cdot4$

$$2d_1=-\quad 15\cdot95$$

$$\therefore \quad d_1=-7''\cdot925$$

$$\theta_1'=66^\circ\ 52',\ \theta_0'=53'$$

$$\therefore \quad \phi_1=\frac{66^\circ\ 52'-53'}{4}=16^\circ\ 30',\ \psi_1=\frac{66^\circ\ 52'+53'}{4}=16^\circ\ 56'$$

also $\phi_2=32^\circ\ 23'$, $\psi_2=32^\circ\ 50'$, and $d_2=-10''\cdot2$.

log 7·925, - -	0·8990
„ sin 32° 23′, -	9·7288
AC log 10·2, - -	8·9914
„ cosec 16° 30′,	0·5467
log β, - - -	0·1659
log cos 32° 50′, -	9·9244
„ sec 16° 56′, -	0·0192
2 log sec x, - -	20·1095
log sec x, - -	10·0547
log cot 16° 56′, -	0·5165
„ tan x, - -	9·7287
„ tan x, - -	9·7287
„ cot y, - -	9·8812
„ cot y, - -	9·8812
„ $\tan\frac{\alpha}{2}$(neg)(=151°25′)	=9·7363

log β, - - -	0·1659
„ sin 32° 50′, - -	9·7342
„ cosec 16° 56′, -	0·5357
2 log sec y, - -	20·4358
log sec y, - - -	10·2179

$$\therefore \quad \frac{\alpha}{2}=151^\circ\ 25'$$

$$\psi_2-\frac{\alpha}{2}=32^\circ\ 50'-151^\circ\ 25'$$

$$=-118^\circ\ 35'$$

$$\therefore \quad e=10\cdot2\ \text{cosec}\ (32^\circ\ 50')\times \sec\ (61^\circ\ 25')$$

log 10·2, - - -	1·0086
„ cosec 32° 23′, -	0·2712
„ sec 61° 25′ - -	0·3202
log $e(=39''\cdot81)$ - -	=1·6000

Hence, taking the arithmetic means of these two equally good determinations we have

$$\frac{a}{2}=149° \ 52',$$
$$e= \ 42 \ \ 02\,,$$

and from these the following table was constructed.

TABLE.

θ'	$e \sin \frac{\theta' - a}{2}$	θ'	$e \sin \frac{\theta' - a}{2}$	θ'	$e \sin \frac{\theta' - a}{2}$
$-10°$	$-17''\cdot87$	50°	$-34''\cdot49$	110°	$-41''\cdot88$
0	21 ·14	60	36 ·39	120	42 ·00
+10	24 ·21	70	38 ·15	130	41 ·86
20	27 ·06	80	39 ·45	140	41 ·36
30	29 ·80	90	40 ·63	150	40 ·56
40	32 ·20	100	41 ·33		

α Geminorum and β Leonis on 14 and 16 March, taking the mean of the two days' observations, March gave

$$\theta_1'=62° \ 6' \text{ and } \theta_0'=53'.$$

From table, - - -	62° 6′ gives	$-36''\cdot8$
,, - - -	0 53 ,,	$-21\ \cdot4$
Correction to reading of sextant, -		$-15\ \cdot4$
Mean of observed distances corrected for all the other errors, - -		60° 40′ 33″
Corrected apparent distance, - -		60 40 17 ·6
Calculated distance, - - - -		60 40 17 ·2

α Geminorum and α Ophiuchi.

$\theta_2'=130° \ 25'$, correction from table, -	$-41''\cdot8$
$\theta_0'=53'$,, ,, -	$-21\ \cdot4$
Correction to reading of sextant,	$-20\ \cdot4$
Mean of observed distances corrected for all errors but the above, -	127° 2′ 27″·75
Correction for centering, - - -	$-20\ \cdot4$
Corrected observed distances, - -	127 2 7 ·35
Calculated ,, - -	127 2 7 ·35

η Ursae Majoris and α Ophiuchi.

$\theta_1' = 66° 52'$, correction from table,	-	$-37''\cdot6$
$\theta_0' = 0\ 53$,, ,,		$-21\cdot4$
Correction for centering, - -	-	$-16\cdot2$

Mean observed distances corrected for all errors but centering, - -	59° 6′ 50″·35
Correction for centering, - - -	−16 ·2
Corrected observed distance, - -	59 6 34 ·15
Calculated apparent distance, - -	59 6 34 ·4

When the observing objects are too bright to be viewed by the naked eye, coloured glass shades are used. These consist of three different sorts:

The first, fitted to the eyepiece of the telescope, darkens equally the direct and reflected images; if glass of such a shade be chosen, free from flaws, as to be capable of transmitting distinct and well defined images of the objects, both the direct and reflected images will be equally displaced at the point of contact by the surfaces of the glass not being exactly parallel, and therefore no error will be introduced into the reading of the instrument when using it to protect the eye. This is the best shade to use unless the direct and reflected images differ too much from each other in brightness, and is exactly suitable for observations made with the mercurial horizon.

The second sort are placed on the frame of the sextant so that they can be interposed between the object and the index glass. Any defect in these displaces the reflected image only.

The third sort are placed on the frame of the sextant so that they can be interposed between the object viewed directly and the horizon glass, and only affect the direct image of the object.

To determine the error arising from using any shade or combination of shades of the two latter sorts, measure the sun's diameter, or other fixed determinate angle, first using the shade applied to the eyepiece of the telescope and then any combination of shades of the two other sorts—with or without an eyepiece shade of a lighter description—the difference between the two results will be the error introduced by the combination used. When a sufficient number of equations are thus formed, the error of each shade will be found by eliminating those of the others.

A sextant is liable to slight alterations of form when held in different positions, which can be found by measuring known angles in different positions of the instrument. In good instruments these are small, and constantly recur in the same position of the instrument. This liability to error should be constantly kept in sight, and the observations to be made with a sextant

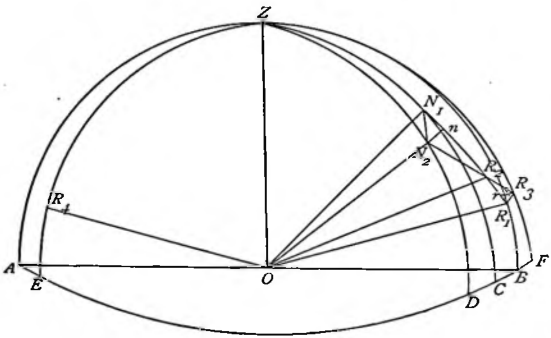


FIG. 7.

should be arranged in pairs, so as to be equally affected by these errors; but in opposite directions, so as to disappear when the mean of the two observations or their results are taken.

The mercurial horizon is a very important instrument for taking observations on shore. Its roof consists of two pieces of glass set in a frame, so that their outer surfaces may be at right angles to each other, or nearly so. It is placed over the mercury trough to protect the mercury from the wind. Each piece of glass ought to be free from flaws, with its surfaces exactly parallel to each other.

The error arising from the surfaces of the glass composing the roof not being exactly parallel to each other, but inclined at a very small angle, may be determined as follows:

Let R_1O be drawn in the plane of the paper parallel to a ray from a heavenly body incident on the outer surface of the roof of a mercurial horizon, and let the surface of a sphere whose centre is O cut the plane of the paper in the great circle of which AZB is the semi-circumference, having its diameter AOB parallel to the surface of the mercury, and the radius OZ perpendicular to it; let ON_1 and ON_2 be drawn parallel respec-

tively to the normals to the outer and inner surfaces of the glass of the roof, but not necessarily in the plane of the paper, to which, for the sake of illustration, we shall suppose them inclined; let N_1 and N_2 be the points in which they meet the surface of the sphere. Let the plane passing through Z and N_1 cut the surface of the sphere in the great circle of which ZN_1C is a quadrant, and the plane passing through Z and N_2 cut it in the great circle of which ZN_2D is a quadrant; join N_1R_1 by the arc of a great circle, and in it take the point R_2 such that $\text{sine}\, N_1R_2 = \frac{\text{sine}\, N_1R_1}{\mu}$, μ being the refractive index of the glass composing the roof; join R_2O, which will be parallel to the direction of the ray on the glass after suffering refraction at its outer surface; join N_2R_2 by the arc of a great circle which produce to R_3, and make N_2R_3 such that sine $N_2R_3 = \mu \sin N_2R_2$; join R_3O, which will be parallel to the direction of the ray in the air after being refracted at the inner surface of the glass; let the plane passing through ZO and R_3 cut the sphere in the great circle of which FR_3ZR_4E is the semicircumference, and make $ZR_4 = ZR_3$; join OR_4, which will be parallel to the direction of the ray after it has been reflected from the surface of the mercury. In the arc $N_1R_2R_1$ take the points n and r such that $R_2n = R_2N_2$ and $R_2r = R_2R_3$; join N_2n and R_3r by arcs of great circles, and let R_3r cut ZR_1 in r'; then Zr' will be very approximately equal to ZR_3, since N_1N_2 is very small, and R_1r' will be the error in the observed altitude arising from the surfaces of the glass composing the side of the roof, which is turned from the observer, not being exactly parallel to each other. To calculate this, let $N_1ON_2 = i$, a very small angle, the square and higher powers of which may be neglected without introducing an error of any practical importance; ZN_1N_2* $= a$, $N_1ZR_1 = \phi$, $CN_1R_1 = \psi_1$, $ZR_1N_1 = \psi_2$, $ZR_1 = z$, $ZN_1 = \frac{\pi}{4}$ very approximately, $N_1n = i\cos(a - \psi_1)$, we have

$\therefore \quad R_2n = R_2N_2;\ N_1R_2 = R_2n + nN_1 = R_2N_2 + i\cos(a - \psi_1).$

Also $\sin N_1R_1 = \mu \sin N_1R_2$,

$$\begin{aligned}
&= \mu \sin\{N_2R_2 + i\cos(a - \psi_1)\}, \\
&= \mu \sin N_2R_2 + \mu \cos N_2R_2 \times i\cos(a - \psi_1), \\
&= \sin N_2R_3 + i\cos(a - \psi_1)\,\mu \cos N_2R_2.
\end{aligned}$$

$$\therefore \quad N_1R_1 = N_2R_3 + i\cos(a - \psi_1)\frac{\mu \cos N_2R_2}{\cos N_2R_3}, \text{ very approx.,}$$

$$= nr + i\cos(a - \psi_1)\frac{\mu \cos N_2R_2}{\cos N_2R_3};$$

* Measured from ZN_1 round to the right.

but $\quad N_1R_1 = R_1r + nr + i\cos(a-\psi_1)$,

$$\therefore \quad R_1r + i\cos(a-\psi_1) = i\cos(a-\psi_1)\frac{\mu\cos N_2R_2}{\cos N_2R_3};$$

$$\therefore \quad R_1r = i\cos(a-\psi_1)\left\{\frac{\mu\cos N_2R_2}{\cos N_2R_3} - 1\right\};$$

but $\quad R_1R_3 = \dfrac{R_1r}{\cos R_1}$ very approximately,

$$= i\frac{\cos(a-\psi_1)}{\cos R_1}\left\{\frac{\mu\cos N_2R_2}{\cos N_2R_3} - 1\right\};$$

or putting in the term multiplied by i

$N_2R_3 = N_1R_1 = z - \dfrac{\pi}{4}$, we have

$$R_1R_3 = i\frac{\cos(a-\psi_1)}{\cos R_1}\left\{\frac{\sqrt{\mu^2 - \sin^2\left(z-\frac{\pi}{4}\right)}}{\cos\left(z-\frac{\pi}{4}\right)} - 1\right\} \text{.......(1)}$$

Angle $ZR_1R_3 = R_1 - \psi_2$,

and $\quad \cos ZR_3 = \cos z \,.\, \cos R_1R_3 + \sin z \,.\, \sin R_1R_3 \cos(R_1 - \psi_2)$

$\qquad = \cos z + R_1R_3 \sin z \,.\, \cos(R_1 - \psi_2)$ very approx.

$$\therefore \quad ZR_3 = z - \frac{i\,.\,\cos(R_1-\psi_2)\,.\,\cos\overline{a-\psi_1}}{\cos R_1}$$

$$\times\left\{\frac{\sqrt{\mu^2 - \sin^2\left(z-\frac{\pi}{4}\right)}}{\cos\left(z-\frac{\pi}{4}\right)} - 1\right\} \text{..................(2)}$$

In the triangle $R_1R_2R_3$ we have

$\tan R_1 = \dfrac{R_3r}{R_1r}$ very approximately,

$R_3r = \sin R_2R_3 \sin R_2$ very approximately,

and $\quad R_1r = i\cos(a-\psi_1)\left\{\dfrac{\mu\cos N_2R_2 - \cos N_2R_3}{\cos N_2R_3}\right\}$

$$= \frac{i\cos(a-\psi_1)}{\cos N_2R_3\,.\,\sin N_2R_2}\sin(N_2R_3 - N_2R_2)$$

$$= \frac{i\cos(a-\psi_1)}{\cos N_2R_3}\cdot\frac{\sin R_2R_3}{\sin N_2R_2};$$

$$\therefore \quad \tan R_1 = \frac{\cos N_2R_3\,.\,\sin N_2R_2\,.\,\sin R_2}{i\cos(a-\psi_1)}$$

$$= \tan(a-\psi_1)\cdot\cos N_2R_3;$$

$$\therefore \quad 1+\tan^2 R_1 = 1+\tan^2(\alpha-\psi_1)\,.\cos^2 N_2R_3$$
$$= 1+\tan^2(\alpha-\psi_1)-\tan^2(\alpha-\psi_1)\sin^2 N_2R_3;$$
$$\therefore \quad \sec^2 R_1 = \sec^2(\alpha-\psi_1)-\tan^2(\alpha-\psi_1)\sin^2 N_2R_3;$$
$$\therefore \quad \frac{\cos(\alpha-\psi_1)}{\cos R_1} = \sqrt{1-\sin^2(\alpha-\psi_1)\sin^2 N_2R_3}$$
$$= \sqrt{1-\sin^2(\alpha-\psi_1)\,.\sin^2\left(z-\frac{\pi}{4}\right)} \text{ very approx....(A)}$$

When ϕ is small and $\left(z-\frac{\pi}{4}\right)$ large, substituting this value in (2) we have

$$ZR_3 = z - i\sqrt{1-\sin^2(\alpha-\psi_1)\sin^2\left(z-\frac{\pi}{4}\right)}\,.\cos(R_1-\psi_2)$$
$$\times\left\{\frac{\sqrt{\mu^2-\sin^2\left(z-\frac{\pi}{4}\right)}}{\cos\left(z-\frac{\pi}{4}\right)}-1\right\}\text{.........(3)}$$

where
$$\left.\begin{aligned}\sin\psi_1 &= \sin\phi\frac{\sin z}{\sin N_1R_1}\\ \sin\psi_2 &= \frac{\sin\phi}{\sqrt{2}\,.\sin N_1R_1}\end{aligned}\right\}\text{.....................(B)}$$

In these equations we cannot substitute $\left(z-\frac{\pi}{4}\right)$ for N_1R_1 unless it is much larger than ϕ, which, as we have before mentioned, should always be kept as small as possible. We see at once from the equation that $\sin N_1R_1$ can never be less than $\frac{\sin\phi}{\sqrt{2}}$.

Denoting the inclination of the normals to the surfaces of the glass composing the side of the roof turned *towards* the observer by i' and α', the inclination of the ray to the vertical, after it has passed through the glass, by z', and the angle corresponding to R_1 by R_1', we shall have

$$ZR_4 = z' - i'\sqrt{1-\sin^2(\alpha'-\psi_1)\sin^2\left(z-\frac{\pi}{4}\right)}$$
$$\times\cos(R_1'-\psi_2)\left\{\frac{\sqrt{\mu^2-\sin^2\left(z-\frac{\pi}{4}\right)}}{\cos\left(z-\frac{\pi}{4}\right)}-1\right\};$$

but $ZR_4 = ZR_3$; therefore equating their values and transposing,

$$z=z'-\left\{i'\cos(R_1'-\psi_2)\sqrt{1-\sin^2(a'-\psi_1)\sin^2\left(z-\frac{\pi}{4}\right)}-i\cos(R_1-\psi_2)\right.$$
$$\left.\times\sqrt{1-\sin^2(a-\psi_1)\sin^2\left(z-\frac{\pi}{4}\right)}\right\}\left\{\frac{\sqrt{\mu^2-\sin^2\left(z-\frac{\pi}{4}\right)}}{\cos\left(z-\frac{\pi}{4}\right)}-1\right\}.$$

The reading of the sextant, if correct, will give twice the altitude, or if z be expressed in degrees $180^\circ-2z$. Instead of which, on account of the above imperfections in the glass roof, the sextant will read $180^\circ-z-z'$, z' being expressed in degrees; consequently, if 2θ be the reading of the sextant in degrees, the correction for the roof to be applied to this reading is

$$\{i'\cos(R_1'-\psi_2)\sqrt{1-\sin^2(a'-\psi_1)\sin^2(45^\circ-\theta)}$$
$$-i\cos(R_1-\psi_2)\sqrt{1-\sin^2(a-\psi_1)\sin^2(45^\circ-\theta)}\}$$
$$\times\left\{\frac{\sqrt{\mu^2-\sin^2(45^\circ-\theta)}}{\cos(45^\circ-\theta)}-1\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where the values of $\cos R_1$ and $\cos R_1'$ will, on referring to equation (A), be given by

$$\cos R_1=\frac{\cos(a-\psi_1)}{\sqrt{1-\sin^2(a-\psi_1)\sin^2(45-\theta)}},$$
$$\cos R_1'=\frac{\cos(a'-\psi_1)}{\sqrt{1-\sin^2(a'-\psi_1)\sin^2(45^\circ-\theta)}}.$$

When therefore ϕ is made so small that it may be neglected, ψ_1 and ψ_2 disappear also from the expression when $\sin(45^\circ-\theta)$ is numerically much larger than $\sin\phi$, in which case we may put

$$\cos R_1\sqrt{1-\sin^2 a\,.\,\sin^2(45^\circ-\theta)}=\cos a,$$

and $$\cos R_1'\sqrt{1-\sin^2 a'\sin^2(45-\theta)}=\cos a',$$

and the correction in (4) will be reduced to

$$(i'\cos a'-i\cos a)\left\{\frac{\sqrt{\mu^2-\sin^2(45^\circ-\theta)}}{\cos(45^\circ-\theta)}-1\right\}\ \ldots\ldots\ldots(5)$$

which expression can be used when the mercurial horizon is so placed that the plane of the meridian is perpendicular to the surfaces of the glasses composing the roof, and circum-meridian altitudes of bodies which differ considerably from 45° are observed.

In this way the roof may be tested and the values of $i\cos a$ and $i'\cos a'$ may be determined as follows. Mark one end of the roof with some letter, A suppose; commence observing the altitude with the A end turned towards the observer. Whilst the observer is reading off the sextant, an assistant should carefully reverse the roof so that the next observation may

be made with the A end of thero of turned from the observer. The roof should be reversed after each contact has been made, and a number of contacts which is a multiple of 4 made.

Put $$F(\theta)=\frac{\sqrt{\mu^2-\sin^2(45^\circ-\theta)}}{\cos(45^\circ-\theta)}-1.$$

If, when A end of roof is turned to the observer the correction to the reading of the sextant is $(i'\cos\alpha'-i\cos\alpha)F(\theta)$, that for the A end turned from the observer will be $=(i\cos\alpha-i'\cos\alpha')F(\theta)$.

If, therefore, the mean of the two meridian altitudes given by the observations with the A end towards the observer be 2θ, and that given by the observations with the other end towards the observer be $2\theta+2d_1$, we shall have $i\cos\alpha-i'\cos\alpha'=\frac{d_1}{F(\theta)}$.

If we now take the glass at the A end of the roof out of its frame, turn it upside down so that the top edge of the glass may be at the bottom and the inner surface the outer one, and replace it in its frame, the sign of $\cos\alpha'$ will be changed, and on repeating the observation on the same body, if $2d_2$ be the difference between the mean of the two circum-meridian altitudes, we shall have

$$i\cos\alpha+i'\cos\alpha'=\frac{d_2}{F(\theta)},$$

and $\therefore$ $$\left.\begin{aligned} i\cos\alpha&=\frac{d_1+d_2}{2F(\theta)}\\ i'\cos\alpha'&=\frac{d_2-d_1}{2F(\theta)}\end{aligned}\right\}\ldots\ldots\ldots\ldots(6)$$

$F(\theta)$ can be determined as follows, taking $\mu=1\cdot51$:

To calculate the expression $\frac{\sqrt{\mu^2-\sin^2(45^\circ-\theta)}}{\cos(45^\circ-\theta)}-1$, we observe it is the same as $\sqrt{\frac{(\mu+1)(\mu-1)}{\cos^2(45^\circ-\theta)}+1}-1=\sec x-1=2\sin^2\frac{x}{2}\sec x$;

where $\tan^2 x=\frac{\overline{\mu+1}\,.\,\overline{\mu-1}}{\cos^2(45^\circ-\theta)}$; take $\mu=1\cdot51$ and $\theta=67^\circ\ 36'$.

$\log\overline{\mu+1}(=2\cdot51)$, -	$0\cdot399674$	$\log\sec x$, - -	$0\cdot199147$
„ $(\mu-1)(=0\cdot51)$, -	$\bar{1}\cdot707570$	„ 2, - - -	$0\cdot301030$
„ sec 22° 36′, -	$0\cdot034699$	„ sin (25° 23′ 30″),	$9\cdot632259$
„ „ -	$0\cdot034699$	„ „ (25° 23′ 45″),	$9\cdot632325$
	2) $0\cdot176642$	log 0·582, - -	$\bar{1}\cdot764761$

log tan x(= 50° 47′ 15), $0\cdot088321$

$\therefore$ $F(67^\circ\ 35')=\cdot582$, $F(45^\circ)=0\cdot51$, and $F(15^\circ)=0\cdot691$.

Therefore a low altitude is the best for determining the value of $i \cos \alpha - i' \cos \alpha'$; also when θ does not differ much from 45°, ψ_1 and ψ_2 will become large angles approaching 90° as their maximum value, and cannot be neglected. Consequently, in making the observations from which equation (6) resulted, either a large or small altitude should be selected, as the two following examples will illustrate:

23 June, 1857, in latitude 45° 20′ N., the following circum-meridian altitudes of the sun were observed with a mercurial horizon, the first contact being made with the end of the roof marked *A* turned towards the observer; whilst the observer read off the sextant, an assistant carefully reversed the roof, so that 10 good contacts were made and read with the *A* end towards the observer, and 10 with the *A* end turned from the observer. The following results were obtained:

2 × meridian alt. of sun's lower limb, mean of 10 contacts, *A* end of roof *towards* observer,	135° 42′ 13″·6
Mean of 10 contacts with *A* end of the roof *from* the observer, - - - - -	135° 42′ 26″·4

The glass on the *A* end of the roof was then taken out of the frame, turned half round on an axis perpendicular to its length, so that the top edge of the glass now became the bottom edge, and the outer surface the inner; and on the 3rd of July following, at the same place, a similar observation gave

2 × meridian alt. of sun's lower limb, mean of 10 contacts, *A* end *towards* observer, - -	134° 43′ 20″·3
Mean of 10 contacts with *A* end *from* the observer, - - - - - - -	134° 43′ 19″·4

Referring to equation (6),

$2d_1 = 12''\cdot8$ or $d_1 = 6''\cdot4$, and $2d_2 = -0''\cdot9$ or $d_2 = -0''\cdot45$, and $\dfrac{\sqrt{\mu^2 - \sin^2(45° - \theta)}}{\cos(45° - \theta)} - 1 = 0\cdot582$ in this case.

$$\therefore \qquad i \cos \alpha = \frac{d_1 + d_2}{1\cdot164} = \frac{5''\cdot95}{1\cdot164} = 5''\cdot1,$$

$$i' \cos \alpha' = \frac{d_2 - d_1}{1\cdot164} = \frac{-6''\cdot85}{1\cdot164} = -5''\cdot9;$$

the second position of the glass of the roof should therefore be retained, and the error to be applied to the double altitude observed with the *A* end towards the observer will be

$$-0''\cdot8\left\{\frac{\sqrt{\mu^2 - \sin^2(45° - \theta)}}{\cos(45° - \theta)} - 1\right\},$$

and will be of no practical importance if the *A* end of the roof is kept always turned towards the observer.

In making these observations the cover of the artificial horizon must be carefully placed, so that the surfaces of the glass composing its roof may be as nearly as possible perpendicular to the plane of the meridian.

July 1866, in latitude 36° 41′ 30″ N., the following circum-meridian altitudes of α Pis. Aust. (Fomalhaut) were observed, and gave the following results:

Mean of 12 contacts with A end of roof *towards* the observer, - -	2 mer. alt. 46° 0′ 32″·4
Mean of 12 contacts with A end *from* the observer, - - - -	2 mer. alt. 46° 0′ 28″·2

Glass of roof on A side reversed, and a few nights after a similar observation on Fomalhaut was made which gave

Mean of 12 contacts, A end *towards* observer, 2 altitude of Fomalhaut, - - - -	46° 0′ 24″·2
Mean of 12 contacts, A end *from* the observer,	46° 0′ 35″·7

Here
$$\frac{\sqrt{(1{\cdot}51)^2-\sin^2(22^\circ)}}{\cos 22^\circ}-1=0{\cdot}578,$$

$$2d_1=-4''{\cdot}2,\ d_1=-2''{\cdot}1, \text{ and } 2d_2=11''{\cdot}5, \text{ or } d_2=5''{\cdot}75\ ;$$

$$\therefore \qquad i\cos\alpha=\frac{-2''{\cdot}1+5''{\cdot}75}{1{\cdot}156}=3''{\cdot}16,$$

$$i'\cos\alpha'=\frac{5''{\cdot}75+2{\cdot}1}{1{\cdot}156}=6{\cdot}8''.$$

Therefore the glass on the A end of the roof should be replaced in its former position, when the correction will be expressed by

$$-3''{\cdot}6\left\{\frac{\sqrt{(1{\cdot}51)^2-\sin^2(45^\circ-\theta)}}{\cos(45^\circ-\theta)}-1\right\}.$$

Hence, when using the mercurial horizon the same end of the glass roof must always be kept turned towards the observer, and the plane of reflection kept as nearly as possible in coincidence with the vertical plane, perpendicular to the surfaces of the glass of the roof; the mercury to be used in the trough must always be kept clean. When dirty it can be quickly cleaned by pouring the mercury from the bottle to the trough, keeping the bottom of the bottle vertically over its mouth, and immediately any drop appears arrest the pouring by placing the finger on the aperture; the mercury in the trough will be bright and clean, with the exception of the small quantity of drop that may have escaped from the bottle before the pouring was stopped; unscrew the bottom from the top, keeping the spout downwards; the drop will be in the top, which should be carefully cleaned. The mercury must then be poured carefully back into the bottle, and the drop

which always keeps on the top of the mercury, or as much of it as possible, retained in the trough and cleaned out of it. This process must be continued until all the drop has disappeared, and the mercury is altogether clear and bright.

To examine a theodolite, select a firm level piece of ground, upon which set up its stand with the ends of the legs placed at the angular points of an equilateral triangle and pressed well into the earth, so that the axis of the screw for securing the instrument to the stand may be as nearly vertical as possible. Screw the theodolite tightly on the stand, turn the levelling screws up evenly, so as to take the weight of the instrument and keep its lower plate clamp firmly attached to the stand; examine the clamp and tangent screw of the lower plate to see that the former holds the plate rigidly in one position when clamped, and that the latter and the plate move simultaneously in either direction. Examine the clamp and tangent screw of the upper plate in a similar manner, and, these proving satisfactory, clamp the upper plate at reading 360°, turn the instrument until the spirit levels on the upper plate are so placed that their lengths may be respectively parallel to the straight lines joining the levelling screws on the opposite sides of the instrument, so that one pair of levelling screws may act on one level without altering the bubble of the other, and *vice versa;* clamp the instrument and level it, then unclamp the upper plate, examine the bubbles of the spirit levels, which should retain their central positions, turn the upper plate slowly round, and clamp it at reading 180°. The bubbles of the levels, which are slightly displaced by the motion of the upper plate, ought to return to their central positions whenever the motion is arrested, and when the upper plate is clamped at 180°. If this is the case it shows that the spirit levels are correct, and the instrument level; if they do not return to their positions, but come to rest in other situations, the upper plate being clamped at reading 180°, measure the distance of each bubble from its proper place, halve the results, and by means of the adjusting screws of the levels bring each bubble half way towards its proper position, and afterwards by means of the levelling screws of the instrument the other half way into its central position; unclamp the upper plate, and turn it to reading 360°. If the adjustment has been properly made, the spirit-level bubbles will come to rest in their central positions. Should this not be the case, the bubbles must again be brought into position half way by the spirit-level adjusting screws and the other half way by the levelling screws, the upper plate turned to

reading 180°, and so on, until the spirit-level bubbles rest in their central positions at both readings of the upper plate.

Adjust the eyepiece of the telescope to its position for distinct vision of the cross wires, and then the field glass to its position for distinct vision of the object. The upper plate being unclamped, place the telescope on its Y's, and secure it firmly there, so that the line of collimation may not be liable to any vibration from a looseness of the telescope in them; turn the upper plate until the cross wires of the telescope intersect a distinct well-defined distant object, clamp the upper plate, and with a slow gentle motion turn the telescope round in its Y's, so that it may revolve about its line of collimation. If the cross of the wires remains on the object in every position of the telescope, it is in its proper place; otherwise it must be brought there by the adjusting screws and firmly secured.

Level the instrument, if necessary, and bring the vertical wire of the cross wires into its proper position by turning the telescope gently round in its Y's, until, by means of the tangent screw of the vertical circle, the wire be made to move up and down over the image of the same point of a distant well-defined object, which it must coincide with through its whole length; being satisfied in this respect, unclamp the vertical circle and bring the bubble of the spirit level under the telescope, as nearly as possible to its central position, by turning the vertical circle by hand so as to elevate or depress the telescope as the case may be, clamp the vertical circle and bring the telescope bubble to its position by means of the tangent screw of the vertical circle; see that the bubbles of all the other levels are in their proper places; all the bubbles being in their right positions, read off the vertical circle, which, if the instrument is in perfect adjustment, should be zero. This will not generally be the case, but will be some small quantity, a suppose, which must be noted. Place a levelling staff as far from the theodolite as possible, so that its zero mark may be distinctly seen through the telescope, raise or lower the staff until the zero mark of the staff *exactly* coincides with the cross of the telescope wires, place a second levelling staff close to the theodolite with its zero mark exactly the same height as the cross of the telescope wires, read off and note the reading of the staff. Remove the theodolite and set it up close to the first levelling staff, and measure the difference between the height of the cross of the telescope wires and the height of the zero mark of the first levelling staff, and make a corresponding alteration in the height of the zero mark of the second levelling staff. By raising or depressing the telescope bring the cross of

the wires into coincidence with the zero mark of the second levelling staff, read off the vertical circle and note it. Suppose this reading to be β, by means of its tangent screw set the vertical circle to reading $\frac{\alpha+\beta}{2}$, which will make the line of collimation horizontal, raise or lower the zero mark of the second levelling staff until it *exactly* coincides with the cross of the telescope wires. The straight line joining the cross of the telescope wires and the zero mark of the second levelling staff will now be horizontal; bring the bubble of the telescope level into its central position by means of its adjusting screws, the cross of the telescope being in coincidence with the zero mark of the levelling staff during the operation, as well as all the other spirit level bubbles in their proper places, and the reading of the vertical circle $\frac{\alpha+\beta}{2}$. In good instruments $\frac{\alpha+\beta}{2}$ will always be very small, but must be carefully noted, and applied to the reading of

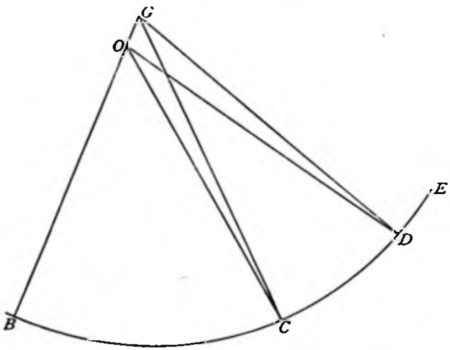


FIG 8.

the vertical circle, so long as the position of the cross wires of the telescope remain unaltered.

The upper plate of a theodolite is subdivided into two or more equal parts, at which points verniers are placed for the purpose of reading the graduated circle on the lower plate; the arithmetical mean of the differences of the readings taken at these equidistant points, corresponding to any two positions of the upper plate, has the important advantage of being free from the error arising from the vertical axis, about which the line of collimation of the telescope turns,

not passing exactly through the centre of the graduated circle.

Let n be the number of equidistant verniers on the upper plate, then $\frac{2\pi}{n}$ will be the distance between the zeros of any two consecutive verniers; let G (Fig. 8) be the centre of the graduated circle $BCDE$, C and D the positions of the m^{th} vernier corresponding to an observed angle A between two objects; O the point in which the axis, about which the upper plate revolves, meets the plane of $BCDE$. Join CO, DO, and GO, which latter produce to meet the circumference CDE in B, of which the reading on the graduated circle is a.

Let the reading of the m^{th} vernier at C be $\frac{2m\pi}{n}+\theta_m$ and that of the same vernier at D be $\frac{2m\pi}{n}+\theta'_m$.

$$\therefore \quad \text{angle } CGB = a - \frac{2m\pi}{n} - \theta_m,$$

$$\text{,,} \quad DGB = a - \frac{2m\pi}{n} - \theta'_m.$$

$$\therefore \quad \text{,,} \quad CGD = \theta'_m - \theta_m.$$

Since GO is very small with respect to GC or OC, $\frac{GO}{OC}$ may be considered equal to a very small quantity e, and therefore we shall have

$$\text{angle } GCO = e\sin(a - \frac{2m\pi}{n} - \theta_m) \text{ very approximately,}$$

$$\text{,,} \quad GDO = e\sin(a - \frac{2m\pi}{n} - \theta'_m) \quad \text{,,}$$

but $$COD + GDO = CGD + GCO$$

$$\therefore\ COD = CGD + GDO - GCO$$
$$= \theta'_m - \theta_m + e\left\{\sin\left(a - \frac{2m\pi}{n} - \theta'_m\right) - \sin\left(a - \frac{2m\pi}{n} - \theta_m\right)\right\}.$$

Each of the other verniers will give a similar result, and by adding them all together we shall have

$$nA = \{\theta'_1 + \dots + \theta'_m + \dots + \theta'_n\} - \{\theta_1 + \dots + \theta_m + \dots + \theta_n\}$$
$$+ e\left\{\sin\left(a - \frac{2\pi}{n} - \theta'_1\right) + \dots + \sin\left(a - \frac{2m\pi}{n} - \theta'_m\right) + \dots\right.$$
$$\left. + \sin(a - 2\pi - \theta'_n)\right\} - e\left\{\sin\left(a - \frac{2\pi}{n} - \theta_1\right) + \dots\right.$$
$$\left. + \sin\left(a - \frac{2m\pi}{n} - \theta_m\right) + \dots + \sin(a - 2\pi - \theta_n)\right\};$$

or written more concisely

$$nA = \Sigma_n(\theta'_m - \theta_m) + e\Sigma_n\left\{\sin\left(\overline{a-\theta'_m} - \frac{2m\pi}{n}\right) - \sin\left(\overline{a-\theta_m} - \frac{2m\pi}{n}\right)\right\} (1)$$

Now $\theta'_1, \ldots \theta'_m \ldots$ and θ'_n will always differ but little from each other, and therefore, from θ' their arithmetic mean, and the same remark applies to θ the arithmetic mean of $\theta_1, \ldots \theta_m \ldots$ and θ_n, and therefore, in the terms multiplied by e we may replace $\theta_1, \ldots \theta'_m \ldots$ and θ'_n by θ', and $\theta_1, \ldots \theta_m \ldots$ and θ_n by θ without introducing any sensible error, and the sum of the terms multiplied by e will be very approximately equal to

$$\Sigma_n\left\{\sin\left(\overline{a-\theta'} - \frac{2m\pi}{n}\right) - \sin\left(\overline{a-\theta} - \frac{2m\pi}{n}\right)\right\}$$

$$= \{\sin(a-\theta') - \sin(a-\theta)\}\Sigma_n\left(\cos\frac{2m\pi}{n}\right)$$

$$+ \{\cos(a-\theta) - \cos(a-\theta')\}\Sigma_n\left(\sin\frac{2m\pi}{n}\right) = 0.$$

Since $\qquad \Sigma_n\left(\cos\frac{2m\pi}{n}\right) = 0$ and $\Sigma_n\left(\sin\frac{2m\pi}{n}\right) = 0$

and equation (1) is reduced to

$$nA = \Sigma_n(\theta'_m - \theta_m),$$

$$\therefore \qquad A = \frac{\Sigma_n(\theta'_m - \theta_m)}{n} = \theta' - \theta \ldots\ldots\ldots\ldots\ldots\ldots \text{Q.E.D.}$$

The verniers are generally denoted by the capital letters A, B, C, etc., and their readings by the A, B, C, etc., readings respectively.

When using the tangent screws, on all occasions, the screw should be kept turning in the *same* direction at the *end* of each operation.

Clamp the upper plate with the zero of the A vernier near 360°, and by means of the tangent screw make the reading exact, read off all the other verniers and note the readings. In the same horizontal plane with the instrument, or nearly so, select two distinct well defined distant objects, making with each other at the theodolite an angle of about 60°; unclamp the lower plate, turn the instrument until the cross of the telescope wires nearly intersects the left hand object; clamp the lower plate, and by means of its tangent screw make the intersection exact; unclamp the upper plate, and turn it from left to right until the cross of the telescope wires nearly intersects the right hand object, clamp the upper plate, and make the intersection exact, read off and note the reading of all the verniers; unclamp the upper

plate, and continuing its motion from left to right, bring the cross wires round to intersect the first object again, read off all the verniers and note the readings. The *A* reading should be now 360°, or differ very slightly therefrom, and the readings of the other verniers should be same as before, or the difference slight. Should this not be the case repeat the operation carefully, and if the want of agreement is found to arise from any instrumental defect, it must be immediately repaired, because the instrument will not be fit for use until it has been done. But if the agreements are sufficiently exact, set the *A* vernier at 120° and take similar rounds of angles between the same two objects. Then set the *A* vernier successively at 180° and 240° and repeat the operations, carefully reading the verniers and noting the readings corresponding to each intersection.

Next, take the right hand object as zero, and still keeping the motion of the telescope from left to right, take a series of similar observations between the same two objects, carefully noting all the readings of each vernier.

The comparison of the different values of the angle thus obtained between the same two objects will afford a means of estimating the degree of accuracy of the instrument, and also of determining the errors arising from mechanical imperfection at different parts of the circumference.

Examine the magnetic needle to see if its north and south extremities are sharp and well defined, that it is well balanced, turns freely on its pivot at the slightest motion of the instrument about its vertical axis, and that the apparatus for raising it off its pivot works well.

When required for use the magnetic needle must be dropped gently on its pivot, from which it must otherwise be kept raised, and care taken when putting the instrument away that the needle has as far as possible its proper position.

The theodolite being levelled and its upper plate clamped with its *A* reading 360°, unclamp the lower plate and turn the instrument until the telescope is perpendicular to the direction of the magnet. It will then be in the best position for reading the ends of the needle; clamp the lower plate, and by means of its tangent screw bring the ends of the needle to readings 90° and 270° on its graduated circle. In doing this, cause the needle to vibrate gently and to come to rest, alternately moving in opposite directions. The line of collimation of the telescope will now be east and west magnetic; move the tangent screw of the upper plate, turning it continuously in one direction until, the magnetic force overcoming the friction of the pivot, the needle commences to return to its normal position, when

cease turning the tangent screw and read off the A vernier. Note the reading, and after the needle has come to rest reverse the motion of the tangent screw, turning it gently in the opposite direction, at the same time carefully watching the extremity of the needle, and the instant it begins to move arrest the motion of the tangent screw, read off, and note the reading of the A vernier. This operation must be repeated several times, alternately turning the tangent screw in opposite directions, and noting the readings of the A vernier. Take the arithmetic mean of the readings corresponding to the motion of the tangent screw from right to left, and that corresponding to the motion of the tangent screw in the opposite direction; the difference between the two results will be twice the angle of deflection of the magnet due to the friction of the pivot, and the half sum of the same two results, E suppose, will be the reading of the upper plate when the line of collimation of the telescope is due east and west magnetic, and the theodolite properly prepared for observing the magnetic bearings of any object; thus unclamp the upper plate and turn the telescope so that the cross wires of the telescope may nearly intersect the object; clamp the upper plate, and make the cross of the wires intersect the object, read off the A vernier, and suppose B be the result, then if E be the reading of the east point as determined above, $E-B$ will be the magnetic bearing of the object measured from the east point, and $90°+E-B$ when measured from the magnetic north towards the east.

The motion of the telescope with the vertical circle of a theodolite causes its line of collimation to trace out a plane in which the axis of rotation of the upper plate should lie or be parallel to, but this is not always the case; they are generally inclined to each other at a small angle, which remains constant unless the instrument receives an injury or is taken to pieces. To determine this inclination a small scale of equal parts must be attached to the spirit level of the upper plate, whose length is parallel to the axis about which the telescope revolves, in order to read the ends of the level bubble. Having done this, level the theodolite carefully, and tighten *all* the screws well up, and when satisfied in this respect the screws must not be touched throughout the whole series of observations; the clamp of the lower plate must not be touched after the zero mark has been set, nor its tangent screw; the clamps of the upper plate and the vertical circle and their tangent screws only are to be touched throughout the series; but the telescope must be referred to the zero mark, and all the verniers carefully read before and after each observation. Take three observations of the sun for true bearing at equal altitudes on

each side of the meridian in the manner to be pointed out in a subsequent chapter, one altitude being as low as possible, another at a large altitude, and the other about half the sum of the other two; the readings of the bubble of the cross level must be carefully taken before and after each observation, whilst the upper plate is clamped. Each pair of equal altitudes will give the theodolite reading of the direction of the meridian, which will include the error arising from the plane of collimation not being vertical; and since the effect of the error varies with the altitude of the observed body, a comparison of the three readings will enable the inclination of the axis of rotation of the upper plate to the plane of collimation, and the value of a division of the scale, to be determined, as well as the true direction of the meridian.

The following will show how this can be done.

Let $ASZB$ (Fig. 9) be the intersection of a vertical plane passing through O, the centre of the horizontal plate of a theodolite, and a heavenly body S, with the celestial concave; aSb the intersection of the celestial concave by the plane described by the line of collimation of the telescope of the

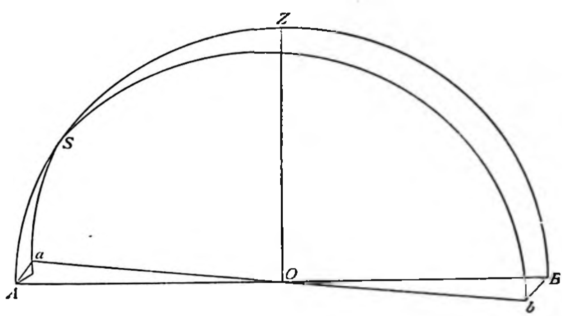


FIG. 9.

theodolite passing through S, Aa and Bb the arcs of the horizon intercepted between the two planes, OZ the straight line joining O and its zenith Z, x the inclination of the axis about which the upper plate revolves to the plane of collimation, a the reading of the bubble of the cross level on the upper plate, and y the value of one division of its scale, x and y being expressed in the same unit.

Hence the angle $ASa = x + ay$.

In the triangle ASa, right angled at A,

$$\tan Aa = \sin AS \tan ASa.$$

Aa and ASa are both very small, and therefore when expressed in circular measure we have

$$Aa = ASa \sin AS.$$

$$\therefore \quad \frac{Aa}{ASa} = \sin AS,$$

where Aa and ASa may be expressed in any unit, so long as the same unit is taken for both.

Now Aa is the error in the theodolite reading, consequent on the error in the position of the plane of collimation passing through S;

$$\therefore \quad \text{error in theodolite reading} = (x + ay) \sin A \ldots\ldots\ldots (1)$$

putting AS the altitude of $S = A$.

Let B_1, B_2, B_3 be three bearings of the meridian derived from observations made with the theodolite at altitudes A_1, A_2, and A_3 of the heavenly body respectively, of which A_1 is as small, and A_3 as large as possible, and $A_2 = \frac{A_1 + A_3}{2}$ or nearly so.

Then if B be the true bearing of the meridian

$$\begin{aligned} B &= B_1 + (x + a_1 y) \sin A_1, \\ &= B_2 + (x + a_2 y) \sin A_2, \\ &= B_3 + (x + a_3 y) \sin A_3, \end{aligned}$$

where a_1, a_2, and a_3 are the respective readings of the bubble of the cross level.

Subtracting the first of the foregoing equations from the second we have

$$O = B_2 - B_1 + x(\sin A_2 - \sin A_1) + y(a_2 \sin A_2 - a_1 \sin A_1);$$

$$\begin{aligned} \therefore \quad x(\sin A_2 - \sin A_1) + y(a_2 \sin A_2 - a_1 \sin A_1) &= B_1 - B_2 \\ &= d_1 \text{ suppose} \ldots (2) \end{aligned}$$

Similarly subtracting the first from the third we have

$$\begin{aligned} x(\sin A_3 - \sin A_1) + y(a_3 \sin A_3 - a_1 \sin A_1) &= B_1 - B_3 \\ &= d_2 \text{ suppose} \ldots (3) \end{aligned}$$

Let $\quad m_1 = \sin A_2 - \sin A_1; \quad m_2 = \sin A_3 - \sin A_1,$

and $\quad n_1 = a_2 \sin A_2 - a_1 \sin A_1; \; n_2 = a_3 \sin A_3 - a_1 \sin A_1;$

and the equations are reduced to

$$m_1 x + n_1 y = d_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$m_2 x + n_2 y = d_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

from which the values of x and y can be quickly and easily determined.

The following observation will show how this is done.

At altitude 5°, the reading of bubble of the cross level of the

theodolite was $-0{\cdot}8$ divisions of the scale, and the reading of the theodolite gave the bearing of the meridian for this altitude $65°\ 29'\ 24''{\cdot}2$.

At altitude $31°\ 30'$, the reading of the cross level bubble was $-1{\cdot}6$ divisions of the scale, and the corresponding reading cf the theodolite for the bearing of the meridian $65°\ 29'\ 44''{\cdot}2$.

At altitude $58°$, the reading of the bubble of the cross level was $-2{\cdot}5$ divisions of the scale, and the reading of the theodolite corresponding to the bearing of the meridian given by this observation $65°\ 30'\ 12''{\cdot}9$.

Referring to equations (2) and (3) we find

$$d_1 = -20''\,;\ d_2 = -48''{\cdot}7,$$
$$m_1 = \sin(31°\ 30') - \sin 5° = 2\sin(13°\ 15')\cos(18°\ 15') = 0{\cdot}4354,$$
$$m_2 = \sin 58° - \sin 5 = 2\sin(26°\ 30')\cos(31°\ 30') = 0{\cdot}7609,$$
$$n_1 = -1{\cdot}6 \times \sin(31°\ 30') + 0{\cdot}8 \times \sin 5° = -0{\cdot}7714,$$
$$n_2 = -2{\cdot}5 \times \sin 58° + 0{\cdot}8 \times \sin 5° = -2{\cdot}0544\,;$$

$\therefore$ substituting these values in (4) and (5),

$$0{\cdot}4354 \times x - 0{\cdot}7714 \times y = -20'' \quad\text{.................(6)}$$
$$0{\cdot}7609 \times x - 2{\cdot}0544 \times y = -48{\cdot}7 \quad\text{................(7)}$$

Multiplying (6) by 2 we have

$$0{\cdot}8708 \times x - 1{\cdot}5428 \times y = -40''.$$

Dividing (6) by 4,

$$0{\cdot}1089 \times x - 0{\cdot}1929 \times y = -5''.$$

Taking the difference of these two,

$$0{\cdot}7619 \times x - 1{\cdot}3499 \times y = -35'',$$

but $$0{\cdot}0010 \times x - 0{\cdot}0019 \times y = -0''{\cdot}05\,;$$

$$\therefore \quad 0{\cdot}7609 \times x - 1{\cdot}348 \ \times y = -34''{\cdot}95.$$

Equation (7) gives

$$0{\cdot}7609 \times x - 2{\cdot}0544 \times y = -48''{\cdot}7\,;$$
$$\therefore \quad 0{\cdot}7064 \times y = -13''{\cdot}75,$$
$$y = \frac{137500''}{7064} = 19''{\cdot}5.$$

Substituting this value of y in (7),

$$0{\cdot}7609 \times x - 40''{\cdot}06 = -48''{\cdot}7\,;$$
$$\therefore \quad x = -\frac{86400''}{7609} = -11''{\cdot}3.$$

$$\therefore \quad B = B_1 - \ 2''{\cdot}2 = 65°\ 29'\ 24''{\cdot}2 - \ 2''{\cdot}2 = 65°\ 29'\ 22'',$$
$$= B_2 - 22''{\cdot}2 = 65°\ 29'\ 44''{\cdot}2 - 22''{\cdot}2 = 65°\ 29'\ 22'',$$
$$= B_3 - 50''{\cdot}9 = 65°\ 30'\ 12''{\cdot}9 - 50''{\cdot}9 = 65°\ 29'\ 22''.$$

If the theodolite is fitted like an altitude and azimuth instrument, with a stride level for its horizontal axis, the error in the level of the horizontal axis in each position can be determined, and then one observation on the heavenly body will be sufficient to determine the direction of the meridian if the stride level be carefully read in both its positions.

CHAPTER III.

LATITUDE.

Having determined the errors of the sextant and artificial horizon in the manner pointed out in the previous chapter, the observer is prepared to take a series of observations for latitude. In doing this, he should always use the same sextant and the same horizon until the series is completed.

To determine the latitude of a place absolutely star observations are the best. The stars for observation should be carefully selected in pairs of nearly equal meridian altitudes, one passing to the north and the other to the south of the zenith. The altitudes of stars nearly equal will be very nearly equally affected by the fluctuating corrections for the errors of the sextant depending on the altitude of the star, and a small inaccuracy in their determination will not make a sensible difference in the corrections to be applied to the observed altitudes; and as these corrections affect the latitudes derived from the meridian altitudes of the two stars in opposite directions, no sensible trace of them will be found in the mean of the two latitudes thus determined. The permanent errors will destroy each other in the results of all north and south observations so meaned, whatever may be the difference between the meridian altitudes of the stars. The readings of the sextant should be kept between 120° and 40°, or the zenith distances of the stars proper for these observations will be between 30° and 70°. We may here observe that when the place of observation is between 20° and 60° north latitude, the pole star can always be used for one of the stars passing north of the zenith.

Two observers should observe together, the one taking the altitudes of the star with his own sextant and artificial horizon, whilst the other takes the time with a sidereal chronometer and notes the time and altitude in the observing book, after taking one pair of stars north and south. The other observer should take the next pair with his own sextant and artificial horizon, whilst the other takes and notes the time for him.

The two artificial horizons should be placed at a convenient distance east and west of each other, with their sides parallel to the meridian, so that the plane of the meridian may be perpendicular to the surfaces of the glass of their roofs, the marked end of which must always be turned towards the observer. The artificial horizon when not in use should be provided with a cover to protect it from the dew.

The stars to be observed, the time of their passing the meridian, and their meridian altitude should be noted in the observation book before commencing the observations, and ample time should be given between each observation to turn the cover of the artificial horizon, and for the observer to change his seat from north to south, or *vice versa.* The barometer and thermometer should be read and noted in the observation book before commencing the night's observations, and after they have been completed observations for index error must be made before and after observing the altitudes, and noted in the observation book, as well as at every convenient opportunity during the intervals between them; but care must be taken not to tire the eye. Each coincidence should be deliberately and well observed, alternately turning the tangent screw in opposite directions, and always taking an even number of coincidences of the same star. Great care must be taken to read off the sextant, with ample time to do it well.

In circum-meridian observations the same number of coincidences should be made on each side of the meridian, so that they can be arranged in pairs of nearly equal altitudes. It is a good plan to use each eye alternately, one to make the coincidence and the other to read off the sextant, and then change eyes, the change being distinguished by placing the letter *r* when the coincidence is made with the right eye, and the letter *l* when made with the left.

The error of collimation should be determined the night before commencing a series of observations for latitude, and after they are completed, if the sextant is carefully handled and carried, the error of collimation ought not to alter sensibly during a series of observations for latitude; but if one is compelled at any time to use large altitudes, the collimation error should be specially determined, unless an equally large altitude can be observed on the other side of the zenith.

The error of the sidereal chronometer can be determined with sufficient accuracy for the calculation of the circum-meridian altitudes by taking half the sum of the times corresponding to equal altitudes of the star on opposite sides of the meridian, derived from the first and last observations, and comparing it with the star's right ascension.

The observer's seat should be so arranged that he can seat himself comfortably and conveniently with respect to the instrument, with the light behind him so that he can turn it on the arc of the sextant when reading it off, the assistant taking the time sitting behind him. When there is only one observer, the light must be so placed as to illuminate the face of the sidereal chronometer, so that the observer can read off and note the time of coincidence *before* he reads off the sextant.

About half an hour before the star to be observed comes to the meridian, set the sextant index at twice its estimated altitude, and if using a stand for the sextant, place it so that the line joining two of the foot screws may be perpendicular to the meridian, and the third foot screw in the meridian and nearest to the mercurial horizon, and so that the observer can see the image of the star reflected from the mercurial horizon through the telescope collar. He then turns the sextant so as to bring its plane vertical, when the image of the star reflected by the index glass will pass near to that seen in the mercurial horizon. He should move the index bar until the two images appear close together, then clamp the index bar, and having adjusted the telescope to distinct vision, with its parallel wires in their proper situation, screw it firmly into the collar. If observing without a stand, which was my usual practice, the observer seats himself so as to see the image of the star reflected from the mercurial horizon in the centre of the trough, or as nearly so as possible, holding the sextant in his hands, with his elbows resting firmly on his knees. He looks through the telescope collar at the image of the star seen in the mercurial horizon, just over the edge of the horizon glass, and turns the sextant round his line of sight, until the image of the star reflected from the index glass passes near it. He unclamps the index bar if necessary, so as to make a closer approach of the two stars, clamps the index, and screws in the telescope as before.

Bring the two images of the star into the middle of the field of view of the telescope, and with the tangent screw bring the two images close, but in such a position that the motion of the star in altitude will bring the two images into coincidence in ten or twelve seconds, this time turning the tangent screw at the *end* of its motion from right to left suppose. The assistant taking the time should now be directed to count the seconds of the sidereal chronometer, whilst the observer by means of the foot screws of the stand, or with a turn of his wrist, makes the image, reflected from the index glass, vibrate across the image reflected from the mercurial horizon, making a complete vibration in a second of time,

so that the moving image passes the other every half second, and thus the time of coincidence can be estimated with great accuracy. In setting the index for the next coincidence the tangent screw at the end of the motion should be turned in the opposite direction, or from left to right. The star should be followed in this manner across the meridian until it has fallen to an altitude as nearly as possible equal to the first observed altitude; the number of altitudes observed should be even, the same number on each side of the meridian, and the bar set by the tangent screw turned alternately in opposite directions. The interval between two successive coincidences should be from eighty to one hundred and twenty seconds of time, but the essential points are exact coincidences regularly taken at nearly equal intervals, careful reading, while the observer's eye must never feel tired, as when it does it is not in a fit state for observing.

To illustrate what has been said we will take some examples of the absolute determination of the latitude of the place of observation.

At Gozo Lighthouse, 23 August 1867, the following observations were made with sextant C, the error of centering of which is given in the following table, and the error of collimation on this night and for the observations on the 27th August, hereafter given, was determined by observations taken before and after the observations were recorded was 11′ 30″, the barometer was 29′·87, and thermometer 78°·5 Fahrenheit.

CORRECTIONS FOR THE ERROR OF CENTERING OF SEXTANT C.

θ'	$e\sin\frac{\theta'-a}{2}$	θ'	$e\sin\frac{\theta'-a}{2}$	θ'	$e\sin\frac{\theta'-a}{2}$	θ'	$e\sin\frac{\theta'-a}{2}$	θ'	$e\sin\frac{\theta'-a}{2}$
−10°	+ 8″·9	20°	+13″·5	50°	+17″·2	80°	+19″·7	110°	+21″·0
0	+10 ·6	30	14 ·9	60	18 ·2	90	20 ·3	120	21 ·0
+10	+12 ·1	40	16 ·1	70	19 ·1	100	20 ·7	130	21 ·0
								140	20 ·7

OBSERVATIONS ON α AQUILÆ near the meridian, taken with sextant C, the marked side of the artificial horizon being turned towards the observer.

R.A. of α Aquilæ $19^h\ 44^m\ 20^s\cdot4$, and Declination $8°\ 31'\ 31''$ N.

	Time by Sidereal Chronometer.	Observed Altitude.	Time from Meridian.	Correction.	2 × Meridian Altitude.
	h m s	° ′ ″	m s	′ ″	° ′ ″
(1)	7 35 8	124 45 10	9 54·5	+11 6·6	124 56 16·6
(2)	36 54	48 50	8 8·5	7 32	56 22
(3)	38 29	51 30	6 33·5	4 52	56 22
(4)	40 21	53 40	4 41·5	2 29	56 9
(5)	41 42	55 0	3 20·5	1 15·8	56 15·8
(6)	42 41	55 30	2 21·5	37·5	56 7·5
(7)	44 10	56 0	52·5	5·2	56 5·2
(8)	45 12	56 5	P.M. 9·5	0	56 5
(9)	46 45	55 40	1 42·5	19·8	55 59·8
(10)	47 49	55 5	2 46·5	52·2	55 57·2
(11)	49 45	53 45	4 42·5	2 30	56 15
(12)	50 50	52 40	5 47·5	3 47·6	56 27·6
(13)	52 39	49 30	7 36·5	6 33	56 3
(14)	53 34	48 0	8 31·5	8 13	56 13
(15)	54 40	45 50	9 37·5	10 28·6	56 18·6
(16)	55 58	42 40	10 55·5	13 30	56 10

Arithmetic mean, - - - - - -	124° 56′ 11″·57
Index error carefully determined by observations before and after observation,	−1 14 ·3
	124° 54′ 57″·4
	62° 27′ 28″·7
Correction for error of centering, taken from table - - - - -	+10 ·6
Collimation error, - - - - -	− 4 ·4
Refraction, - - - - - -	−28 ·7
Meridian altitude, - - - - -	62° 27 6″·2
Zenith distance, - - - - - -	27° 32 53″·8 N.
Declination, - - - - - -	8 31 31 N.
Latitude by α Aquilæ, - - - -	36° 4 24″·8 N.

We first determine the chronometer time of the star's meridian passage by means of the first and two last observa-

tions; thus the first altitude observed is intermediate in value between the two last, and from these two we determine the time. When after passing the meridian the altitude of α Aquilæ was 124° 45′ 10″, from the three last observations we see that from chronometer time $53^m\ 34^s$ to $54^m\ 40^s$, the altitude diminished 130″, and thence to $55^m\ 58^s$ it diminished 190″. Hence in first interval of 66^s the altitude diminished 130″, whilst in the second interval of 78 it diminished 190″, hence

during the first interval mean diminution of altitude in $1^s = 2''$
second „ „ „ $= 2 \cdot 4$

We therefore suppose the altitude at chronometer time $7^h\ 54^m\ 40^s$ to be diminishing at the rate of $2''\cdot 2$ in 1^s of time, and that it moved 40″ uniformly at this rate, in which case the chronometer time of the observed altitude changing from 124° 45′ 10″ to 124° 45′ 50″ will be 17^s, or nearly so; adding this to $7^h\ 54^m\ 40^s$ we have the time of the P.M. altitude 124° 45′ 50″ to be $7^h\ 54^m\ 57^s$; the A.M. time of the same altitude was $7^h\ 35^m\ 8^s$. The half sum of these, or $7^h\ 45^m\ 2^s\cdot 5$, gives the time of the star's meridian passage sufficiently near for the purpose of calculating the corrections to be applied to the observed altitude to obtain the star's meridian altitude, which is done as follows:—

The difference between the time of observation and the time of the star's meridian passage, as given by the *sidereal* chronometer, gives the time from meridian inserted in the third column of the tabulated observations, which is the hour angle of the star.

Rule.—To the logarithm 0·60206 add the log cosine of the latitude of the place, the log cosine of the star's declination, and the log secant of half the greatest reading of the sextant; the sum will give a logarithm which is constant for the same place and star; to this logarithm add twice the log sine of half the star's hour angle; the sum will be the log sine of the correction, which added to the corresponding reading of the sextant will give its reading for the meridian altitude of the star.

In making these calculations four places in the logarithms will generally be sufficient.

Constant log, - - - - - -	0·6021
Log cosine lat. place (36° 4′), - - -	9·9076
„ „ star's declination (8° 31′·5), -	9·9952
„ secant (62° 28′), - - - - - -	0·3350
Log constant for Gozo lighthouse and star α Aquilæ, - - - - - -	0·8399

(1) observation in table, chron. time, -	$7^h\ 35^m\ 8^s$
Time of α Aquilæ's M.P. by chron., -	7 45 2·5
α Aquilæ's hour angle at No. (1) obs.,	9 54·5
Half hour angle, - - - - -	$4^m\ 57^s{\cdot}25$
Constant log, - - - - - - -	0·8399
Log sine $4^m\ 57^s{\cdot}25$, - - - - - -	8·3348
" " - - - - - -	8·3347
Log sine correction (11′ 6″·6), - - -	7·5094

In a similar manner the other corrections in the table were obtained.

To determine the value of the result of one set of observations as compared with others, for the purpose of meaning the whole, we proceed as follows: Take the difference between the result of each observation, and the arithmetic mean of the whole composing the set, and add together the differences thus obtained irrespective of sign, and let s represent the sum derived from n observations; then we take $\pm\frac{s}{n}$ to represent the most probable error of the result of each observation of the set, and $\pm\frac{s}{n^2}$ the most probable value of the error of the arithmetic mean of the results of the set. The value of a set as compared with others is taken to be inversely as its probable error, or as $\frac{n^2}{s}$.

The arithmetic mean of the results in the foregoing table is 124° 56′ 11″·7. Subtracting this from each of the results we find

(1) gives - - - - - -	4″·9
(2) " - - - - - -	10·3
(3) " - - - - - -	10·3
(4) " - - - - - -	2·7
(5) " - - - - - -	4·1
(6) " - - - - - -	4·2
(7) " - - - - - -	6·5
(8) " - - - - - -	6·7
(9) " - - - - - -	11·9
(10) " - - - - - -	14·5
(11) " - - - - - -	3·3
(12) " - - - - - -	15·9

(13) gives - - - - - -	8·7
(14) " - - - - - -	1·3
(15) " - - - - - -	6·9
(16) " - - - - - -	1·7
$s=$	113″·9

Hence probable error of one observation $= \pm\frac{113\cdot9}{16} = \pm 7''\cdot1.$

" " arithmetic mean $= \pm\frac{7''\cdot1}{16} = \pm 0''\cdot44.$

Value for meaning with others $\frac{16\times16}{113\cdot9} = \frac{256}{113\cdot9} = 2\cdot25$, very appr.

The cover of the artificial horizon was reversed after the observations on α Aquilæ were finished, and the same observer with the same sextant, assisted by the same person taking time with the same sidereal chronometer, took the following observations on α Cephei, whose right ascension was $21^h\ 15^m\ 28^s\cdot5$, and declination 62° 1′ 41″·5. The time of α Cephei's M.P., determined in the same way as the preceding from the first observation, combined with the two last, was $9^h\ 16^m\ 10^s$, and from it the corrections given in the fourth column of the following table were calculated.

α CEPHEI.

Time by Sidereal Chronometer.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Altitude.
h m s	° ′ ″	m s	′ ″	° ′ ″
(1) 9 2 32	127 56 45	13 38	+10 32	128 7 17
(2) 3 56	58 20	12 14	8 31	6 51
(3) 6 41	128 1 55	9 29	5 4	6 59
(4) 8 1	3 25	8 9	3 45	7 10
(5) 9 18	4 30	6 52	2 40	7 10
(6) 10 31	5 25	5 39	1 49	7 14
(7) 11 47	5 55	4 23	1 5	7 0
(8) 16 2	7 0	0 8	0	7 0
(9) 17 24	6 45	1 14	5	6 50
(10) 20 26	5 45	4 16	1 2	6 47
(11) 21 37	5 10	5 27	1 41	6 51
(12) 22 59	4 10	6 49	2 38	6 48
(13) 24 54	2 55	8 44	4 20	7 15
(14) 26 39	0 30	10 29	6 14	6 44
(15) 29 2	127 57 50	12 52	9 23	7 13
(16) 30 22	56 0	14 12	11 17	7 17

Arithmetic mean, - - - - - -	128° 7′ 1″·6	
Index error, - - - - - - -	−1 14 ·3	
	128° 5′ 47″·3	
	64° 2′ 53″·6	
Correction for sextant's error of centering,	+10 ·6	
Collimation error, - - - - - -	− 4 ·8	
Refraction, - - - - - - -	−26 ·8	
Meridian altitude, - - - - - -	64° 2′ 32″·6	
„ zenith distance, - - - -	25° 57′ 27″·48	
Declination, - - - - - - -	62 1 41 ·5	N.
Latitude by α Cephei north of zenith, -	36° 4′ 14″·1	N.
„ α Aquilæ south „ -	36 4 24 ·8	N.
Mean, free from instrumental and personal errors, - - - - - - -	36° 4′ 19″·45	N.

The latitude of Gozo Lighthouse, determined from these combined with numerous other observations made by the same and other observers, was 36° 4′ 16″·4 N.

On 27 August 1867, the same observers with the same sextant, artificial horizon, and watch, made the following observations on the same stars at Spencer's Monument, Malta, the error of collimation being the same as before, barometer being 29·75 and thermometer 75° Fahr. The error of the sidereal watch was 31ˢ slow.

α AQUILÆ, RIGHT ASCENSION $19^h\ 44^m\ 20^s{\cdot}3$.

Time by Chronometer.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Altitude.
h m s	° ′ ″	m s	′ ″	° ′ ″
(1) 7 34 15	125 8 30	9 31	+10 26	125 18 56
(2) 35 37	10 40	8 12	7 40	18 20
(3) 36 37	12 35	7 12	5 55	18 30
(4) 37 30	14 5	6 19	4 33	18 38
(5) 38 17	15 30	5 32	3 30	19 0
(6) 39 55	17 20	3 54	1 45	19 5
(7) 41 32	18 0	2 17	36	18 36
(8) 42 51	18 30	A.M. 0 58	6	18 36
(9) 44 35	18 50	P.M. 0 26	1	18 51
(10) 45 52	18 10	2 3	29	18 39
(11) 47 57	16 35	4 8	1 57	18 32
(12) 49 15	15 40	5 26	3 22	19 2
(13) 50 12	13 50	6 23	4 38	18 28
(14) 51 31	11 55	7 42	6 46	18 41
(15) 53 49	7 40	10 0	11 24	19 4
(16) 55 0	4 50	11 11	14 16	19 6

Arithmetic mean, - - - - - -	125° 18′ 45″·
	−1 7 ·4
2 × Meridian altitude, - - - -	125° 17′ 37″·6
	62° 38′ 48″·8
Correction for error of centering, - -	+10 ·6
Collimation error, - - - - - -	− 4 ·4
Refraction, - - - - - - -	−28 ·5
Meridian altitude, - - - - - -	62° 38′ 26″·5
,, zenith distance, - - - - -	27° 21′ 33″·5
Star's declination, - - - - - -	8 31 31 ·5
Latitude of Spencer's Monument, - -	35° 53′ 5″ N.

The cover of the artificial horizon was then reversed, and the observer shifted his seat to the south side of the horizon, and observed α Cephei as follows:

α CEPHEI, RIGHT ASCENSION $21^h\ 15^m\ 28^s{\cdot}5$.

	Sidereal Chronometer.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Altitude.
	h m s	° ′ ″	m s	′ ″	° ′ ″
(1)	9 3 19	127 36 30	11 38	+ 7 39	127 44 9
(2)	4 50	38 35	10 7	5 44	44 19
(3)	6 19	40 0	8 38	4 13	44 13
(4)	7 22	40 55	7 35	3 15	44 8
(5)	8 45	42 0	6 12	2 10	44 10
(6)	10 9	43 5	4 48	1 18	44 23
(7)	11 50	43 50	3 7	0 33	44 23
(8)	13 49	44 20	1 8	0 4	44 24
(9)	15 17	44 15	P.M. 20	0 0	44 15
(10)	17 2	43 50	2 5	0 15	44 5
(11)	18 10	43 40	3 13	0 35	44 15
(12)	20 20	42 30	5 23	1 38	44 8
(13)	22 27	41 20	7 30	3 11	44 31
(14)	24 20	39 15	9 23	4 58	44 13
(15)	25 30	37 40	10 33	6 17	43 57
(16)	26 35	36 20	11 38	7 39	43 59

Arithmetic mean, - - - - - -	127° 44′ 13″·2
Index error determined by 32 coincidences made in 4 sets of 8 each, carefully read—1 set before observing α Aquilæ, 2 sets after do.; 3 sets before α Cephei, and 4 sets after do., - - - - - -	−1 7 ·4
	127° 43′ 5″·8
	63° 51′ 32″·9
Correction error of centering, - - -	+10 ·6
„ „ collimation, - - -	− 4 ·6
Refraction, - - - - - - -	−27 ·1
Meridian altitude, - - - - -	63° 51′ 11″·8
„ zenith distance, - - - -	26° 8′ 48″·2
α Cephei's declination, - - - - -	62 1 43 N.
Latitude Spencer's Monument, - - -	35° 52′ 54″·8 N.
„ „ „ by α Aquilæ,	35 53 5 N.
Mean, clear of instrumental and personal errors, - - - - - - -	35° 52′ 59″·9 N.

Latitude finally adopted from the mean of numerous observations made by the same and two other observers was 35° 52′ 58″ N.

Suppose when we landed the errors of the sextant had been unknown, only we had been satisfied that the sextant was a good one and its errors not large, we should have had

2 × Meridian altitude of α Aquilæ given by the sextant readings, - - - -	125° 18′ 45″
Meridian altitude by same, - - -	62° 39′ 22″·5
„ zenith distance, - - - -	37° 20′ 37″·5
α Aquilæ's declination, - - - -	8 31 31 ·5 N.
Uncorrected latitude, - - - -	35° 52′ 9″ N.
2 × Meridian altitude of α Cephei given by the sextant readings, - - - -	127° 44′ 13″·2
Meridian altitude by same, - - -	63° 52′ 6″·6

Meridian zenith distance, - - - - -	$26° \; 7' \; 53''.4$
α Cephei's declination, - - - - -	$62 \;\; 1 \;\; 53$ N.
Uncorrected latitude, - - - - -	$35° \; 53' \; 49''.6$ N.
Mean of North and South observation, eliminating constant errors, and very approximately the fluctuating errors, leaving only the difference of refraction $0''.7$ to be applied, - - - - -	$35° \; 52' \; 59''.3$ N.
	$+0 \; .7$
Latitude in which only the difference of the fluctuating errors remains, - - -	$35° \; 53' \;\; 0''$ N.

Hence for general practical purposes with a good sextant the errors of which are not large, by taking the altitudes N. and S. as nearly equal as possible, we can obtain very good results even without knowing the errors of the sextant, and far better than by meaning the results of unequal altitudes taken at hazard without consideration, even if the instrumental errors had been carefully ascertained.

The following is a good example.—Whilst on a sounding voyage in H.M.S. Hydra in the Indian Ocean, having a good opportunity of determining the latitude of an old surveying mark on the island of Hallaneyah, I sent two officers to make the observations described hereafter, having previously arranged the observations and determined the error of the sidereal chronometer,

15 February 1868, at Hallaneyah observation beacon. Barometer $30^{in}.02$ and Thermometer 72° Fahrenheit. Sextant A index error determined by the mean of 24 coincidences carefully taken at 3 separate times of 8 each was $-7''.3$, and that of sextant B, by the other observer, determined in the same way, was $-14''.2$.

One officer, with sextant A, observed Canopus, South, and the Pole Star, North, reversing the roof of the artificial horizon in the manner before described; the other officer, with sextant B, observed ϵ Canis Majoris, South, and α Ursæ Majoris, North, in a similar manner.

CANOPUS, observed with Sextant A, M.P. by Chronometer $1^h\ 58^m\ 7^s$.

	Sidereal Chronometer.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Altitude.
	h m s	° ′ ″	m s	′ ″	° ′ ″
(1)	1 50 1	39 46 20	8 6	+2 36	39 48 56
(2)	51 4	47 5	7 3	2 0	49 5
(3)	52 0	47 35	6 7	1 30	49 5
(4)	53 26	48 30	4 41	0 53	49 23
(5)	54 59	48 45	3 8	0 24	49 9
(6)	56 12	48 55	1 55	0 9	49 4
(7)	57 17	48 55	0 50	0 2	48 57
(8)	59 51	48 55	1 44	0 7	49 2
(9)	2 2 25	48 10	4 18	0 45	48 55
(10)	3 46	47 55	5 39	1 17	49 12
(11)	4 46	47 30	6 37	1 47	49 17
(12)	7 9	45 40	9 12	3 17	48 57

Arithmetic mean, - - - -	39° 49′ 5″·1
Index error, - - - - -	−7 ·3
Twice meridian altitude, - -	39° 48′ 57″·8
Meridian altitude, - - -	19° 54′ 28″·9
Refraction, - - - - -	−2 35 ·6
	19° 51′ 53″·3
Zenith distance, - - - -	70° 8′ 6″·7 N.
Declination, - - - - -	52 37 52 S.
Latitude by Canopus, passing South,	17° 30′ 14″·7 N.

Polaris was then observed, two sets of six each, by same observer with same sextant, the means of which gave

	Sidereal Time.	Sextant Reading.	Sidereal Time.	Sextant Reading.
	$7^h\ 19^m\ 48^s$	35° 0′ 13″·3	$7^h\ 29^m\ 4^s$	34° 53′ 36″·9
Index error, -		−7 ·3	- -	−7 ·3
		35° 0′ 6″		34° 53′ 29″·6
Alt. Polaris, -		17° 30′ 3″	- -	17° 26′ 44″·8
Refraction, -		−2 58 ·7	- -	−2 59 ·3
		17° 27′ 4″·3		17° 23′ 45″·5

From Nautical Almanac, 1 cor.,	+3′ 36″	- -	+6′ 59″
„ 2 cor.,	+0 19	- -	+0 19
„ 3 cor.,	+0 48	- -	+0 47
	−1 0		−1 0
Latitude, - -	17° 30′ 47″·3 N.	- -	17° 30′ 50″·5 N.
	17 30 50 ·5		
Mean by Polaris North of zenith,	17° 30′ 48″·9 N.		

Hence latitude by first observer, A sextant, Canopus passing South, - - -	17° 30′ 14″·7 N.
Latitude by first observer, A sextant, Polaris passing North, - - - - - -	17 30 48 ·9 N.
Mean, eliminating constant errors and very approximately the fluctuating errors, -	17° 30′ 31″·8 N.

The index error might have been omitted, but it should always be observed.

The other observer, with sextant B, observed the following:

ϵ Canis Majoris, with sextant B, Chronometer time M.P. 2h 30m 32s.

	Sidereal Chronometer.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Altitude.
	h m s	° ′ ″	m s	′ ″	° ′ ″
(1)	2 21 31	87 19 10	9 1	+6 9	87 25 19
(2)	22 43	21 5	7 49	4 37	25 42
(3)	24 2	22 15	6 30	3 12	25 27
(4)	25 42	24 5	4 50	1 46	25 51
(5)	26 41	24 30	3 51	1 7	25 37
(6)	27 43	25 0	2 49	0 36	25 36
(7)	30 12	25 25	A.M. 0 20	0 2	25 27
(8)	32 41	25 0	2 9	0 21	25 21
(9)	35 9	23 20	4 37	1 37	24 57
(10)	36 28	23 0	5 56	2 40	25 40
(11)	37 38	21 35	7 6	3 49	25 24
(12)	38 54	20 10	8 22	5 40	25 50

Arithmetic mean, - - - -	87° 25′ 31″
Index error, - - - - -	−14 ·2
2 × Meridian altitude, - - -	87° 25′ 16″·8
Meridian altitude, - - -	43° 42′ 38″·4
Refraction, - - - - -	−1 0
	43° 41′ 38″·4
Meridian zenith distance, - -	46° 18′ 21″·6 N.
Declination, - - - - -	28 48 1 N.
Latitude by ε Canis Majoris S. of zenith, - - - - -	17° 30′ 20″·6 N.

The same observers, with the same sextant, reversed the cover of the artificial horizon and observed α Ursæ Majoris passing to the North of the zenith.

α URSÆ MAJORIS, observed with Sextant B, Chronometer Time M.P. 6^h 32^m 35^s.

	Sidereal Chronometer.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Altitude.
	h m s	° ′ ″	m s	′ ″	° ′ ″
(1)	6 23 28	90 4 45	9 7	+3 24	90 8 9
(2)	25 3	6 15	7 32	2 19	8 34
(3)	26 38	6 45	5 57	1 59	8 44
(4)	27 50	7 50	4 45	0 55	8 45
(5)	29 21	8 0	3 14	0 26	8 26
(6)	30 53	8 20	1 42	0 7	8 27
(7)	32 5	8 20	0 30	0 1	8 21
(8)	33 18	8 20	0 43	0 1	8 21
(9)	35 1	8 10	2 26	0 14·5	8 24·5
(10)	36 52	7 40	4 17	0 45	8 25
(11)	38 1	6 55	5 26	1 12·5	8 7·5
(12)	39 39	6 20	7 4	2 2	8 22
(13)	41 7	5 15	8 32	2 59	8 14
(14)	43 42	3 10	11 7	5 2	8 12

Arithmetic mean, - - - - - -	90° 8′ 23″·7
Index correction, - - - - - -	−14 ·2
	90° 8′ 9″·5
Meridian altitude, - - - - - -	45° 4′ 4″·75
Refraction, - - - - - - -	−56 ·25
	45° 3′ 8″·5
Meridian zenith distance, - - - - -	44° 56′ 51″·5 S.
Declination, - - - - - - -	62 27 34 N.
Latitude by *α* Ursæ Majoris, North, - -	17° 30′ 42″·5 N.
„ *ε* Canis Majoris, South, - -	17 30 20 ·6 N.
Mean, eliminating constant errors and very approximately the fluctuating errors, -	17° 30′ 31″·5 N.
Latitude by other observer with sextant A, using Canopus and Polaris, - -	17 30 31 ·8 N.

This shows that two observers, using different sextants, and each taking stars of nearly equal altitude north and south, will, if good observers, arrive at almost identical results by making the errors on each of the pair they take destroy one another; and I recommend this plan to the reader as fully compensating for the trouble of thinking over and arranging the observations for latitude in this manner before taking them.

The *difference of latitude* between two places can be determined with great accuracy by comparing the meridian altitudes of the same star, corrected for index error, refraction, and the change in the star's declination, observed in the manner before described, by the same observer, with the same instruments, under similar circumstances, supposing the difference of latitude does not exceed two degrees, and the interval of time between the observations is not long, the sextant being a good one for maintaining its errors, which should have been made as small as possible.

Referring to the foregoing observations taken at Gozo and Malta in August 1867, we see that the meridian altitude of *a* Aquilæ at Gozo Lighthouse on the 23rd August was, when uncorrected for refraction and the fluctuating errors of the instruments,-

instruments,- - - - - - - -	62° 27′ 28″·7
Refraction, - - - - - - - - -	−28 ·7
Meridian altitude, 23 August, - - -	62° 27′ 0″
Change of *a* Aquilæ's declination in 4 days, -	+0 ·5
Meridian altitude at Gozo Lighthouse, 27 August,	62° 27′ 0″·5

Meridian alt. α Aquilæ at Spencer's Monument, Malta, on the 27th August, corrected for refraction, - - - - - - -	62° 38′ 20″·3
Difference of latitude, Gozo Lighthouse, north of Spencer's Monument, - - - -	11′ 19″·8
By same observer, with same instrument, α Cephei, 23 August, - - - - -	64° 2′ 26″·8
Change of α Cephei's declination in 4 days, -	−1 ·5
Meridian altitude of α Cephei at Gozo Lighthouse, 27 August, - - - - - -	64° 2′ 25″·3
Meridian altitude of α Cephei at Malta, 27 August, - - - - - - - -	63 51 5 ·8
Difference of latitude, Gozo Lighthouse, north of Spencer's Monument, Malta, - -	11′ 19″·5
Mean of the two determinations, - -	11′ 19″·65
The difference between the two latitudes determined absolutely, as seen before, is - -	11′ 19 ·55

That ultimately arrived at from numerous observations and observers, carefully considered and meaned, was 11′ 18″·4.

The sun affords a very excellent mode of determining differences of latitude by using its motion of declination judiciously, so that the sextant readings at the two places of observation may be as nearly as possible the same. In the following examples the same observer, with the same instruments, under exactly similar circumstances as to shades, artificial horizon, etc., made the observations on the sun's lower or nearer limbs for the purpose of determining the difference of latitude between the two places of observation. The index error of the sextant was determined from careful observations of the sun's diameter on and off the arc. The number of contacts of the limbs before and after the observations were together equal to the number of altitudes observed.

At Point Lepreaux Observation Station, near the Lighthouse, the following observations on the sun's nearer limbs were made on the 8th October 1855, the chronometer time of apparent noon being $11^h\ 55^m\ 19^s$, and the index error of the sextant, with the same shades used, was $+9''$; and the corrections in the table were calculated in the same way, as already described for the stars.

SUN'S NEARER LIMBS OBSERVED, 8 October 1855.

Chronometer Time.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Altitude, ☉.
h m s	° ′ ″	m s	′ ″	° ′ ″
(1) 11 45 50	77 36 30	9 29	+5 19	77 41 49
(2) 47 0	37 30	8 19	4 6	41 36
(3) 48 40	39 10	6 39	2 38	41 48
(4) 49 50	40 20	5 29	1 47	42 7
(5) 51 18	41 5	4 1	0 58	42 3
(6) 52 26	41 45	2 43	30	42 15
(7) 53 52	41 55	1 27	8	42 3
(8) 55 10	42 15	9	0	42 15
(9) 56 20	42 0	1 1	4	42 4
(10) 57 30	41 50	2 11	17	42 7
(11) 58 30	41 25	3 11	37	42 2
(12) 59 44	41 0	4 25	1 9	42 9
(13) 1 30	39 40	6 11	2 17	41 57
(14) 3 12	38 20	7 57	3 49	42 9
(15) 4 16	37 5	8 57	4 44	41 49

Arithmetic mean, - - - - -	77° 42′ 1″
Index error, - - - - - -	+9
	77° 42′ 10″
Meridian altitude, sun's LL., - -	38 51 5
Sun's semi-diameter, - - - -	+16 3
Refraction, less parallax, - - -	−1 6
Meridian altitude of sun's centre, -	39° 6′ 2″
Sun's declination, - - - - -	5 50 29 S.
Meridian altitude of sun's centre when his declination is zero, - -	44° 56′ 31″

Probable size of the error of one observation $\pm 8''\cdot 6$; and the value of the mean for comparison with others $= \frac{15}{8\cdot 6}$.

At the same place the same observer, with the same instruments, exactly under similar circumstances, on the 11th October, made the following observations. The index error of the sextant was found then to be zero, and the chronometer time of apparent noon $11^h\ 56^m\ 40^s$.

SUN'S NEARER LIMBS OBSERVED, 11 October 1855.

Chronometer Time.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Alt. Sun's LL.
h m s	° ′ ″	m s	m s	° ′ ″
(1) 11 49 8	75 22 5	7 32	+3 8	75 25 23
(2) 50 10	23 5	6 30	2 27	32
(3) 51 8	23 55	5 32	1 46	41
(4) 52 5	24 10	4 35	1 14	24
(5) 53 5	24 30	3 35	45	15
(6) 54 0	25 0	2 40	23	23
(7) 54 56	25 10	1 44	11	21
(8) 55 46	25 15	0 54	3	18
(9) 56 46	25 15	0 6	0	15
(10) 57 40	25 10	1 0	4	14
(11) 58 30	25 0	1 50	12	12
(12) 59 55	24 40	3 15	37	17
(13) 12 0 40	24 5	4 0	56	1
(14) 2 25	23 10	5 45	1 56	6
(15) 3 40	22 20	7 0	2 50	10
(16) 4 26	21 45	7 46	3 30	15

Arithmetic mean, - - - -	75° 25′ 18″
Meridian altitude, sun's LL., - -	37° 42′ 39″
Sun's semi-diameter, - - - -	+16 4
Refraction, less parallax, - - -	−1 9
Meridian altitude, sun's centre, - -	37° 57′ 34″
Sun's declination, - - - - -	6 58 58 S.
Meridian altitude of sun's centre when his declination is zero, - -	44° 56′ 32″

The probable size of the error of one of these observations is ±7″, and the value of the arithmetic mean, when comparing this set of observations with others, is therefore $=\frac{16}{7}$. The observation of the 8th gave 44° 56′ 31″, the value of which was $\frac{15}{8{\cdot}6}$; the difference of 1″ must therefore be divided in proportion of $\frac{16\times 8{\cdot}6}{15\times 7+16\times 8{\cdot}6}=0''{\cdot}56$.

The most probable value of the mean of these two sets is ∴ 44° 56′ 31″·56.

After the first of the two foregoing observations at Point Lepreaux the vessel proceeded to Machias Seal Island, and on the 9th October, 1855, the following observations, at the

astronomical station near the Eastern Lighthouse, were taken by the same observer, using the same instruments, under the same circumstances. The time of apparent noon by the chronometer was $11^h\ 58^m\ 5^s$, and the index error of the sextant, carefully determined with the shades used, was $+4''$.

SUN'S NEARER LIMBS OBSERVED, 9 October 1855.

Chronometer Time.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Altitude.
h m s	° ′ ″	m s	′ ″	° ′ ″
(1) 11 49 24	77 58 35	8 41	+4 30	78 3 5
(2) 50 30	59 25	7 35	3 27	2 52
(3) 51 42	78 0 35	6 23	2 26	3 1
(4) 52 44	1 5	5 21	1 42	2 47
(5) 53 36	1 55	4 29	1 13	3 8
(6) 54 30	2 15	3 35	0 47	3 2
(7) 55 20	2 45	2 45	0 28	3 13
(8) 56 30	2 50	1 35	0 8	2 58
(9) 57 30	3 10	0 35	0 1	3 11
(10) 58 30	3 0	0 25	0 1	3 1
(11) 59 26	2 50	1 21	0 8	2 58
(12) 0 50	2 30	2 45	0 28	2 58
(13) 2 10	2 10	4 5	1 0	3 10
(14) 3 24	1 15	5 19	1 42	2 57
(15) 5 0	0 10	6 55	2 54	3 4
(16) 6 20	77 58 50	8 15	4 15	3 5

Arithmetic mean, - - - - -	78° 3′ 2″
Index error, - - - - -	+4
	78° 3′ 6″
	39° 1′ 33″
Sun's semi-diameter, - - - -	+16 3
Refraction, less parallax, - - -	−1 5
Meridian altitude, sun's centre, - -	39° 16′ 31″
Sun's declination, - - - -	6 13 26 S.
Meridian altitude of sun's centre when declination is zero, - - -	45° 29′ 57″

The most probable size of the error of one observation of this set is $\pm 5''\cdot 4$, and the value of the set for meaning with others is $\frac{16}{5\cdot 4}$.

On the 10th of October, or the day following, at the same place, the same observer, under exactly similar circumstances, made the following observations. The chronometer time of apparent noon was $11^h\ 58^m\ 45^s$, and the index error of the sextant, determined in the same way and with the same number of contacts as the preceding, was $+4''$.

SUN'S NEARER LIMBS OBSERVED, 10 October 1855.

	Chronometer Time.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Alt. Sun's LL.
	h m s	° ′ ″	m s	′ ″	° ′ ″
(1)	11 50 0	77 12 40	8 45	+4 34	77 17 14
(2)	51 10	14 0	7 35	3 25	25
(3)	52 28	14 55	6 17	2 21	16
(4)	53 40	16 5	5 5	1 32	37
(5)	55 0	16 30	3 45	0 50	20
(6)	56 12	17 4	2 33	0 24	28
(7)	57 30	17 10	1 15	0 6	16
(8)	58 30	17 10	0 15	0 0	10
(9)	59 40	17 10	0 55	0 3	13
(10)	12 1 0	17 4	2 15	0 18	22
(11)	2 0	16 40	3 15	0 39	19
(12)	3 10	16 15	4 25	1 9	24
(13)	4 16	15 40	5 31	1 51	31
(14)	7 15	13 15	8 30	4 18	33

Arithmetic mean, - - - - -	77° 17′ 22″
Index error, - - - - - -	+4
	77° 17′ 26″
Meridian altitude, sun's LL., - -	38° 38′ 43″
Sun's semi-diameter, - - - -	+16 4
Refraction, less parallax, - - - -	−1 6
Meridian altitude of sun's centre, -	38° 53′ 41″
Sun's declination, - - - - -	6 36 15 S.
Sun's meridian altitude when declination zero, - - - - - -	45° 29′ 56″

The most probable size of the error of one observation of this set is $\pm 6''\cdot 6$, and the value of the set for meaning with the other is $\therefore \frac{14}{6\cdot 6}$; we observe that the difference of the two sets

is 1″, and that we must therefore add to the latter determination

$$\frac{\frac{16''}{54}}{\frac{16}{54}+\frac{14}{88}}=\frac{88''}{151}=0''\cdot58.$$

∴ Meridian altitude of the sun's centre when on the equator at Machias Seal Island Observation Station, - - - -	45° 29′ 56″·58
Do., at Observation Station Point Lepreaux,	44 56 31 ·56
Difference of latitude Lepreaux Station north of Machias Seal Island Station, - -	33′ 25″

which differed from that ultimately arrived at from many other observations combined with it by 1″.

We will give another set of similar observations made to determine the difference of latitude between the astronomical station at Digby, Nova Scotia, and that at Spencer's Anchorage; in this case, as in the former, the observations were all made by the same observer with the same instruments, and the index error determined in the manner recommended in the foregoing pages.

At the astronomical station, Digby, Nova Scotia, on the 25th September, 1856, the following observations were taken, the sextant having no index error, and the chronometer time of apparent noon being $11^h\ 49^m\ 33^s$.

SUN'S NEARER LIMBS OBSERVED, 25 September 1856.

Chronometer Time.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Alt. Sun's LL.
h m s	° ′ ″	m s	′ ″	° ′ ″
(1) 11 40 40	87 50 0	8 53	+5 8	87 55 8
(2) 44 56	53 50	4 37	1 22	12
(3) 46 0	54 25	3 33	0 49	14
(4) 47 4	54 50	2 29	0 24	14
(5) 49 0	55 0	0 33	0 1	1
(6) 49 55	55 15	0 22	0 0·5	15·5
(7) 51 0	55 0	1 27	0 8	8
(8) 52 0	55 10	2 27	0 24	34
(9) 53 4	54 30	3 31	0 48	18
(10) 54 10	54 0	4 37	1 22	22
(11) 55 10	53 15	5 37	2 3	18
(12) 56 20	52 26	6 47	2 59	25
(13) 57 14	51 10	7 41	3 50	0
(14) 58 20	50 0	8 47	5 1	1

Arithmetic mean, - - - - - -	87° 55′ 13″·5
Index error, - - - - - - -	0
Meridian altitude of sun's lower limb, - -	43° 57′ 36″·8
Sun's semi-diameter, - - - - - -	+ 16 0
Refraction, less parallax, - - - - -	− 54 ·4
Meridian altitude of sun's centre, - -	44° 12′ 42″·4
Sun's declination, - - - - - -	1 5 44 S.
Meridian altitude when sun's centre is on the equator, - - - - - - -	45° 18′ 26″·4

SUN'S NEARER LIMBS OBSERVED AT DIGBY, 2 October 1856.

	Chronometer Time.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Alt. Sun's LL.
	h m s	° ′ ″	m s	′ ″	° ′ ″
(1)	11 42 28	82 25 40	6 9	+2 21	82 28 1
(2)	43 50	26 50	4 47	1 24	14
(3)	44 56	27 35	3 41	53	28
(4)	46 20	27 45	2 17	20	5
(5)	47 20	28 10	1 17	7	17
(6)	48 14	28 15	0 23	0·5	15·5
(7)	49 6	28 10	29	1	11
(8)	50 0	28 20	1 23	8	28
(9)	50 50	28 15	2 13	19	34
(10)	51 34	27 55	2 57	33	28
(11)	52 30	27 15	3 53	57	12
(12)	53 36	26 45	4 59	1 32	17

Arithmetic mean, - - - - - - - 82° 28′ 18″

The index error of the sextant, determined in the manner recommended, on the 2nd of October was −12″, and the chronometer time of apparent noon $11^h\ 48^m\ 37^s$.

The arithmetic mean of twice the meridian altitude of the sun's lower limb as given by the sextant readings corrected, - - - - -	82° 28′ 18″
Index error, - - - - - - - -	−12
	82° 28′ 6″

Meridian altitude of sun's lower limb, - -	41° 14′ 3″
Sun's semi-diameter, - - - - - -	+16 2
Refraction, less parallax, - - - - - -	− 1 0
Meridian altitude of sun's centre, - - -	41° 29′ 5″
Sun's declination, - - - - - - -	3 49 21
Sun's meridian altitude when on the equator, -	45° 18′ 26″

Meaning this with the observations made on the 25th September, according to their respective values, we find that the sun's meridian altitude when on the equator is

45° 18′ 26″·25

On the 26th and 29th September the same observer, with the same instruments, under similar circumstances, made the following observations at the astronomical station, Spencer's Anchorage; on the 26th, the index error of the sextant determined from observations then taken was −10″, and the chronometer time of apparent noon on that day was $11^h\ 45^m{\cdot}36^s$. On the 29th September the index error of the sextant was found to be −10″, and the chronometer time of apparent noon was $11^h\ 45^m\ 12^s$.

SUN'S NEARER LIMBS OBSERVED AT SPENCER'S ANCHORAGE, 26 September 1856.

Chronometer Time.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Alt. Sun's LL.
h m s	° ′ ″	m s	′ ″	° ′ ″
(1) 11 37 40	85 48 0	7 56	+3 58	85 51 58
(2) 39 0	48 50	6 36	2 45	35
(3) 40 16	50 10	5 20	1 48	58
(4) 41 18	50 45	4 18	0 55	40
(5) 42 52	51 20	2 44	0 28	48
(6) 44 0	51 24	1 36	0 9	33
(7) 45 10	51 45	0 26	0 0·5	45·5
(8) 46 10	51 20	0 34	0 1	21
(9) 47 20	51 20	1 44	0 13	33
(10) 48 48	50 40	3 12	0 39	19
(11) 50 0	50 20	4 24	1 14	34
(12) 51 20	49 25	5 44	2 5	30
(13) 52 34	48 35	6 58	3 3	38
(14) 53 44	47 30	8 8	4 10	40

Arithmetic mean, - - - - - - -	85° 51′ 38″
Index error, - - - - - - - -	−10
	85° 51′ 28″
Meridian altitude of sun's lower limb, - -	42° 55′ 44″
Sun's semi-diameter, - - - - - -	+16 0
Refraction, less parallax, - - - - -	−56
Meridian altitude of sun's centre, - - -	43° 10′ 48″
Sun's declination, - - - - - - -	1 29 5 S.
Meridian altitude of sun's centre when on the equator, - - - - - - - -	44° 39′ 53″

SUN'S NEARER LIMBS OBSERVED AT SPENCER'S ANCHORAGE, 29 September 1856.

Chronometer Time.	Sextant Reading.	Hour Angle.	Correction.	2 × Meridian Alt. Sun's LL.
h m s	° ′ ″	m s	′ ″	° ′ ″
(1) 11 38 36	83 28 20	6 36	+2 42	83 31 2
(2) 40 6	29 45	5 6	1 37	22
(3) 41 26	30 15	3 46	0 52	7
(4) 43 0	31 0	2 12	0 19	19
(5) 44 0	31 5	1 12	0 6	11
(6) 45 0	31 30	0 12	0 0	30
(7) 46 0	31 5	0 48	0 3	8
(8) 46 45	31 15	1 33	0 9	24
(9) 47 30	31 0	2 18	0 20	20
(10) 48 16	30 55	3 4	0 35	30
(11) 49 10	30 25	3 58	0 58	23
(12) 50 50	29 30	5 38	1 58	28
(13) 51 44	28 30	6 32	2 38	8
(14) 52 30	28 5	7 18	3 18	23

Arithmetic mean, - - - - - - -	83° 31′ 18″
Index error, - - - - - - - -	−10
	83° 31′ 8″
Meridian altitude of sun's lower limb, - -	41° 45′ 34″
Sun's semi-diameter, - - - - - -	+16 1
Refraction, less parallax, - - - - -	−59

Meridian altitude of sun's centre, - - -	42° 0′ 36″
Sun's declination, - - - - - - -	2 39 17 S.
Meridian altitude of sun's centre when on the equator, - - - - - - - -	44° 39′ 53″
Meridian altitude of sun's centre by observations of 26 September, - - - - - -	44 39 53
Mean of 28 observations at Spencer's Anchorage,	44° 39′ 53″
Meridian altitude of sun's centre at Digby Observation Station, mean of 26 observations,	45 18 26 ·25
Spencer's Anchorage, north of Digby, - -	38′ 33″·25

The difference of latitude ultimately adopted was 38′ 33″. From this it follows that judicious observations of the sun's limbs carefully taken with a good sextant give very good differences of latitude.

CHAPTER IV.

DIFFERENCE OF LONGITUDE.

THE general, and frequently the only available, mode of determining the difference of longitude between two places, is by carrying chronometers forward and backward between them, and comparing the chronometers with the time at the astronomical stations, determined from observations made at them with sextants and artificial horizons.

If, however, the surveyor has the good fortune to possess two four-feet portable transit instruments of good make, more accurate results may be obtained, especially if the two astronomical stations can be connected by a telegraph wire; but this seldom, if ever, happens to British surveyors. In fact, during the whole time I was employed only one opportunity of determining a difference of longitude in this manner occurred, and then I was obliged to borrow a four-feet portable transit.

When the former mode is adopted, the chronometers to be carried in the vessel must be selected with great care. They ought not to be sensitive to changes of temperature, and must keep even regular rates. A few good chronometers are far preferable to many indifferent ones. The chronometers should be stowed in a box conveniently situated for winding up and comparing them, in a part of the vessel most free from vibration. A case must be placed outside the chronometer stand, with a space between the two all round so as to perfectly protect the chronometers. After the chronometers have been satisfactorily stowed, they should not be touched, except to be wound. Within the protecting case another box or stand should be placed, in which the chronometers to be carried for observing with should be kept. Four of these are necessary—two set to mean time and two to sidereal time. Small chronometers are best for this purpose, as they generally carry better than the large ones. A thermometer must be placed inside the protecting case in a convenient position for reading.

One of the chronometers which has the smallest rate and

goes most regularly must be selected for a standard, and all the other chronometers must be compared with it forward and backward daily immediately after they are wound, when the thermometer reading should be noted. This should be done *exactly* at the same hour.

When running meridian distances, great care must be taken that the chronometers are wound daily by the *same* person, in the same order, and at the same time, or as nearly so as possible, in order to keep the chronometers running on the same part of their chain. The comparisons of each chronometer with the standard must be made by the *same* persons at the *same* time every day, so that the interval between two consecutive comparisons of the same chronometer with the standard may be twenty-four hours, or very nearly so.

Experience having pointed out the season of the year best adapted for astronomical observation—when the temperature is most even and the fairest weather is to be expected—that season must be selected for running meridian distances. This is a very important point, and must be well considered.

When it is necessary to carry a chronometer for observing with backward and forward between the ship and the observing station, for which purpose pocket chronometers are preferable, they must be carefully compared with the standard chronometer before leaving the ship for observations, and again immediately after returning to the ship from making the observations. These comparisons will show whether the carrying chronometer is going well relatively to the standard. If it does not, it must be rejected and another chronometer used in its place.

The astronomical stations must be prepared for observing—a good sheltered place where the mercurial horizons will not be disturbed by wind, made as near as possible to each of them—and when these and all other necessary arrangements are completed, equal altitudes of the sun must be observed with sextants and artificial horizons in order to determine the errors of the chronometers in the following manner.

At a convenient time in the morning, so as to give the observers ample time to reach the observing station in time to take the first selected altitude of the sun, two observers compare the carrying chronometer with the standard, noting the comparisons in the observing book. One carries the chronometer with great care, and the other the sextant. When the observers reach the station, one places the mercurial horizon in a good position sheltered from the wind, the marked end of its roof being turned towards the observer, and the whole so placed that at the middle of the first observation the vertical plane

passing through the observer and the sun may be perpendicular to the surfaces of the glass roof, or as nearly so as possible. Before each succeeding altitude is observed, the direction of the trough and its roof require a slight change, so as to follow the alteration of the sun's bearing. The other observer, having seated himself in a good position with respect to the mercurial horizon for taking the first altitude of the sun, adjusts the telescope of the sextant to distinct vision, with a coloured eye-piece suitable to the brightness of the sun, and screws it into the telescope socket. He then sets the index bar of the sextant to the selected altitude of the sun with very great care, and clamps it firmly. He then directs the telescope so as to see the image of the sun reflected by the mercurial horizon in the centre of its field of view. Bringing the plane of the sextant vertical, the image of the sun reflected from the index of the sextant will enter the field of view, and the two images of the sun will appear to approach each other. Their nearer limbs must be kept as nearly as possible equi-distant from the centre of the field of view. If the two images are not equally bright, they must be made so by turning the screw which alters the distance of the telescope from the plane of the sextant, then make the image reflected from the index glass vibrate across the nearest point of the limb of the other, so as to make a complete vibration in a second of time. About 15 seconds before the motion of the images will bring their limbs in contact, the observer with the chronometer should be directed to count the seconds of the chronometer, which he must then do evenly and regularly. The instant the limbs touch is estimated by the observer between two successive numbers, as they are counted by the assistant, with great exactness, and when determined by him must be noted by the other in the observation book. The two images of the sun will then overlap and approach coincidence, the instant of which must be determined and noted in the same manner. About 15 seconds before the images separate, the assistant should again count the seconds of the observing chronometer, and the instant of the images separating is estimated and noted as before described. The observer then unclamps the index bar of the sextant, and moves it on one and a half or two degrees, according to the rate of the sun's motion in altitude, taking care to give ample time for setting the index very accurately, and to bring the sextant into its observing position from 20 to 30 seconds of time before the limbs come into contact. He then takes a second observation in the same manner as the first. After three sets have been observed, the observers should change instruments and take more sets in the same way, and if time permits, which it

generally will when the sun is in a good position for such observations, each of the observers may take three more sets; but they must be very careful not to tire their eyes, as a few well observed contacts are far better than many hastily taken, and continued after the eye becomes so tired as to be unfit for good observing.

By this method three times are taken for each setting of the index bar, and therefore plenty of time must be taken to set it accurately. Young observers frequently find the coincidence of the two images of the sun more difficult to determine, and not so accurate as the contact of the limbs, in which case the time may be rejected from the mean, or less weight given to it in meaning with the other two; but they should always be taken and noted, and will, after a little practice, give as good results as the others.

Immediately on reaching the ship after the forenoon observations are finished, the carrying chronometer must be carefully compared with the standard. The time for leaving the ship in the afternoon in order to observe the equal altitudes of the sun in the reverse order must be settled so as to give plenty of time to arrive at the astronomical station, set up the instruments, and be ready for observing the greatest altitude of the sun first, and then the other sets in reverse order. In making these observations it is very important that the observers and their instruments should be similarly circumstanced to that they had in the forenoon, and great attention paid to setting the index of the sextant so as to be exactly in the same position it had at the forenoon observation. We will now take the following example by way of illustration.

At Cape Passaro Castle, on the 1st August, 1866, the equal altitudes of the sun given below were observed.

Sextant Reading.	Chronometer Time, A.M.	Chronometer Time, P.M.	Sum of the Times.
	h m s	h m s	h m s
64° Upper Limb,	6 1 4·5	2 28 36	20 29 40·5
64° Coincidence,	2 23	27 17·5	40·5
64° Lower Limb,	3 42	25 59	41

Mean, - - - - - $20^h\ 29^m\ 40^s{\cdot}67$

Other observations made by the same observer, with the same instrument under similar circumstances, on the same day, gave also—

For sextant reading	66°,	mean sum of times		20^h 29^m	$41^s{\cdot}8$
„	„	68	„	„	41·3
„	„	70	„	„	41·8
„	„	82	„	„	42·8
„	„	84	„	„	42·8

If the sun had not moved in declination during the interval between the A.M. and P.M. observations half of the above sums would have been the chronometer time of apparent noon, and the error of the chronometer on apparent time would immediately result; but the sun, except when it is in the solstices, moves in declination, consequently a correction must be applied to the half sum of the times to give the time of the chronometer at apparent noon. This correction is called the equation of equal altitude and can be calculated in the following manner. The latitude of the observing station at Cape Passaro Castle is, when taken to the nearest minute, 36° 41′ N, and the sun's declination at Cape Passaro, apparent noon, was 18° 2′ N, also taken to the nearest minute; the sun's hourly motion in declination, carefully taken from the Nautical Almanack, was 37″·75.

The half interval between the chronometer times corresponding to sextant reading $64° = 4^h\ 12^m\ 27^s$, the change in the sun's declination in this interval was 159″; hence we have

Latitude,	36° 41′ N.			
☉ Decln.,	18 2 N.	Change of 200″ gives in log cosec a change of 136.		
Diff. - -	18° 39′			
☉ Z. Dist.,	58 0			
Sum, - -	76° 39′ - -	−200″		
Diff., - -	39 21 - -	+200		
½ Sum, -	38° 19′·5 - -	−100″	- log sine - -	−266
½ Diff., -	19 40·5 - -	+100	- - - -	+589

∴ Change of 200″ in the sun's declination gives in twice the log sine of half the sun's hour angle a change of +459

when the sun's altitude is 32° and hour angle $4^h\ 12^m\ 27^s$; but a change of 1^s in this hour angle gives a change of 51·4 in twice the log sine of half the hour angle, and therefore the change of 200″ in the sun's declination will give a change of $\frac{459}{51\cdot4}{}^s = 8^s{\cdot}93$ in the hour angle of the sun at altitude 32°; therefore the change of 159″ gave a change of $7^s{\cdot}1$ in the hour

angle of the sun; the sun was moving from the elevated pole and therefore arrived at the sextant reading 64° in the P.M. observation $14^s{\cdot}2$ sooner than it would have done had the sun's declination remained the same at both the A.M. and P.M. observations. The sum of the times was therefore $14^s{\cdot}2$ less than it would have been had there been no motion in declination, and consequently $7^s{\cdot}1$ must be added to the half sum of the times, to give the chronometer time of apparent noon, or the equation of equal altitudes for sextant reading 64° was $+7^s{\cdot}1$. In a similar manner the equation of equal altitudes corresponding to sextant reading 84° was calculated and found to be $+6^s{\cdot}1$. Hence a change of 20° in the sextant reading gave 1^s change in the equation of equal altitudes. Without sensible error we may assume that the change in the equation of equal altitudes was proportional to the change in the sextant readings, and thus we have

For sextant	reading	64°	equation of	equal altitude	$+7^s{\cdot}1$
„	„	66	„	„	7·0
„	„	68	„	„	6·9
„	„	70	„	„	6·8
„	„	82	„	„	6·2
„	„	84	„	„	6·1

Applying these corrections, we find as follows:

Sextant Reading.	Half Sum of Chronometer Times.	Equation of Equal Alt.	Chronometer fast of Apparent Noon.
	h m s	s	h m s
64°	10 14 50·3	+7·1	10 14 57·4
66	50·9	7·0	57·9
68	50·6	6·9	57·5
70	50·9	6·8	57·7
82	51·4	6·2	57·6
84	51·4	6·1	57·5

Arithmetic mean, - - - - - - -	$10^h\ 14^m\ 57^s{\cdot}6$
Equation of time, - - - - - - -	−6 2·6
Chronometer fast of mean noon, 1 Aug. 1866, -	$10^h\ 8^m\ 55^s$
Chronometer fast of standard chronometer, -	2 14 49·3
Standard fast of Cape Passaro Castle M.T. at noon, 1 August 1866, - - - - -	$7^h\ 54^m\ 5^s{\cdot}7$
Chronometer A fast of standard, - - -	3 30 44·6
Do. do. Cape Passaro M.T. at noon 1 Aug.,	$11^h\ 24^m\ 50^s{\cdot}3$

Two chronometers will be quite sufficient to show the *modus operandi.*

The mean of six sets of equal altitudes of the sun observed at San Ranieri Tower, Messina, on the 2nd August, 1866, by the same observers, with the same instruments, exactly in the same manner as the preceding, gave

Observing chronometer fast of San Ranieri mean time, at noon, 2 August 1866, - - -	$10^h\ 7^m\ 9^s{\cdot}7$
Do. do. standard chronometer, - -	2 14 46
Standard fast of San Ranieri M.T. at noon, 2 August 1866, - - - - - -	$7^h\ 52^m\ 23^s{\cdot}7$
Chronometer A fast of standard, - - -	3 30 50
Do. do. San Ranieri M.T., 2 Aug. 1866,	$11^h\ 23^m\ 13^s{\cdot}7$

At San Ranieri, on the 8th August, 1866, the same observers, with the same instruments, under similar circumstances, observed six sets of equal altitudes of the sun, from the mean of which the following errors of the chronometers were obtained:

Observing chronometer fast of San Ranieri M.T. at noon, 8 August 1866, - - - -	$10^h\ 6^m\ 40^s{\cdot}4$
Do. do. standard chronometer, - -	2 14 21·2
Standard fast of San Ranieri M.T. 8 Aug. 1866,	$7^h\ 52^m\ 19^s{\cdot}2$
Chronometer A, fast of standard, - - -	3 31 22·1
Do. do. San Ranieri M.T., 8 August 1866, -	$11^h\ 23^m\ 41^s{\cdot}3$

After taking these observations the vessel proceeded back again to Cape Passaro Castle, and on the 9th August the same observers, under exactly similar circumstances, observed six sets of equal altitudes of the sun, the mean of which gave as follows:

Observing chronometer fast of M.T. Cape Passaro Castle, at noon, 9 August, - -	$10^h\ 8^m\ 18^s$
Do. do. standard, - - - -	2 14 17·4
Standard chronometer fast of M.T. Cape Passaro Castle, 9 August 1866, - - - - -	$7^h\ 54^m\ 0^s{\cdot}6$
A, fast of standard, - - - - - -	3 31 27·5
„ M.T. Cape Passaro Castle, 9 August, -	$11^h\ 25^m\ 28^s{\cdot}1$

Supposing the passage rates of the chronometers had remained the same during the day's run from Cape Passaro to Messina, and during the return voyage of one day from Messina to Cape Passaro, to determine the meridian distance between the two places. From the half sum of the two errors of each chronometer on Cape Passaro Castle mean time subtract the half sum of the errors of the same chronometer on San Ranieri mean time, and the difference will give the meridian distance between the two places, thus—

Standard chronometer fast of M.T. Cape Passaro Castle, at noon, 1 August 1866, - -	$7^h\ 54^m\ 5^s{\cdot}7$
Same chronometer, fast of M.T., 9 August, -	$7\ \ 54\ \ 0{\cdot}6$
Half sum for Cape Passaro Castle, - - -	$7^h\ 54^m\ 3^s{\cdot}15$

Standard chronometer fast of M.T. San Ranieri, 2 August 1866, - - - - - -	$7^h\ 52^m\ 23^s{\cdot}7$
Do. do. 8 August, - - - -	$7\ \ 52\ \ 19{\cdot}2$
Half sum for San Ranieri, - - - -	$7^h\ 52^m\ 21^s{\cdot}45$

Subtracting this from the former, we find that the standard chronometer gives Cape Passaro Castle $1^m\ 41^s{\cdot}7$ west of San Ranieri.

By chronometer A we have

Half sum of A's error at San Ranieri, - -	$11^h\ 23^m\ 27^s{\cdot}5$
Do. do. Cape Passaro Castle, -	$11\ \ 25\ \ 9{\cdot}2$
Cape Passaro Castle West of San Ranieri, -	$1^m\ 41^s{\cdot}7$

This exact agreement shows that the passage rates of these chronometers on the return voyage were exactly the same as they were on the passage from Cape Passaro to San Ranieri, or they must both have changed exactly to the same amount and in the same direction, which is not probable.

To explain the truth of the above rule, let P_1 denote the error of any one of the chronometers on Cape Passaro Castle mean time given by the observations on the 1st of August, P_2 the error of the same chronometer given by the observations made at Cape Passaro Castle on the 9th August; R_1 the error of the same chronometer on mean time at San Ranieri given by the observations made on 2nd August, and R_2 its error by the 8th August observations; then if r be the

daily passage rate of the chronometer, supposed the same for both passages, we have

$$\text{Meridian distance going} \quad = P_1 + r - R_1,$$
$$\text{,,} \qquad \text{,,} \qquad \text{returning} = P_2 - r + R_2.$$

Eliminating r between these two, we have

$$\text{Meridian distance} = \tfrac{1}{2}\{\overline{P_1+P_2} - \overline{R_1+R_2}\}.$$

Suppose, however, that the daily passage rate on the return voyage had changed from r to $r+e$, the meridian distance would have been $= \tfrac{1}{2}\{P_1+P_2-(R_1+R_2+e)\}$, and the error in the meridian distance, determined on the first assumption, would have been $-\frac{e}{2}$.

We will now take the general case of running meridian distances with chronometers carried between two places, which we will denote by A and B. Suppose by observations made at A, one of the chronometers was found to be a_1 hours, minutes, and seconds fast of mean time; the vessel then leaves A and proceeds to B, where, by observations made n_1 days afterwards, the same chronometer was found to be b_1 hours, minutes, and seconds fast of mean time at B; by subsequent observations at B its error was found to be b_2. The vessel then returned to A, where, from observations made n_2 days after the second set of observations were made at B, the error of the same chronometer on A's mean time was a_2. Let x be the daily rate of the chronometer during the passage from A to B, and we will suppose it to retain the same rate during the return passage from B to A, then if y be the meridian distance of A west of B, we have

$$\text{By first voyage } y = a_1 + n_1 x - b_1,$$
$$\text{,, return ,, } \quad y = a_2 + n_2 x - b_2.$$

Eliminating x we find

$$y = \frac{n_2 a_1 + n_1 a_2 - (n_2 b_1 + n_1 b_2)}{n_1 + n_2}.$$

The accuracy of which, as far as the chronometers are concerned, depends entirely on that of our assumption that the chronometer kept the same daily rate during the return voyage that it had during the passage from A to B. Supposing this not true, but during the return voyage from B to A the daily rate of the chronometer had changed from x to $x+\delta x$, the correct value of y would have been $= \frac{n_2 a_1 + n_1 a_2 - (n_2 b_1 + n_1 b_2)}{n_1 + n_2} - \frac{n_1 n_2 \delta x}{n_1 + n_2}$, and therefore the error arising from our incorrect assumption

is $-\frac{n_1 n_2 \delta x}{n_1+n_2}$; consequently the smaller $\frac{n_1 n_2}{n_1+n_2}$ can be made, the smaller will be the error in the meridian distance from this cause. Now n_1 and n_2 are both positive integers, and $\frac{n_1 n_2}{n_1+n_2}$ has its smallest possible value when $n_1=n_2=1$ and $\frac{n_1 n_2}{n_1+n_2}=\frac{1}{2}$; therefore the best interval between the observations made at the two places is one day. The nearer the number of days in the two passage intervals are to each other the better, and the more like the weather during the passages, and other circumstances under which they are made, are to each other, the smaller the probable size of δx; therefore the vessel should only remain at B about thirty-two hours, just long enough to take equal altitudes of the sun on two consecutive days, and immediately after return to A, so as to arrive in time for observing the equal altitudes of the sun there on the following day.

We have no means of determining the value of δx exactly, but a probable value of it for each chronometer may be found by determining the harbour rates of the chronometers at A before leaving, and immediately after returning thereto. From the rates of the chronometers so determined, their daily comparisons and the temperatures, we may establish a probable relation between the passage rates on the voyages from A to B and from B to A. Suppose we have q chronometers, and the respective changes in these rates on the return voyage to have been δx_1, δx_2 ... and δx_q respectively, and let $p=\frac{n_1 n_2}{n_1+n_1}$; hence on the return voyage No. 1 chronometer will be $n_2\delta x_1$ faster on the mean time of A than if the chronometer had not changed its rate, and the resulting error in the meridian distance will be $p\delta x$; too small if A be to the eastward of B, and too large if to the westward.

Let m be the correct meridian distance between A and B, and m_1 that given by chronometer No. 1. Then for No. 1 chronometer, supposing A to the eastward of B, we shall have

$$m=m_1+p\delta x_1.$$

Supposing m_1 to be the meridian distance given by standard chronometer with which all the others are compared, and δx_2 to be the change in the rate of No. 2 chronometer and c_2 the error arising from comparing it with No. 1, then we shall have, adopting a similar notation,

$$m=m_2+p\delta x_2+c_2,$$

and similarly for the other chronometers,

$$m = m_3 + p\delta x_3 + c_3,$$
$$\text{etc.} = \text{etc.}$$
$$m = m_q + p\delta x_q + c_q.$$

Supposing these equally valuable, and taking their arithmetic mean, we have

$$m = \frac{m_1 + m_2 + \ldots + m_q}{q} + \frac{p}{q}\{\delta x_1 + \ldots + \delta x_q\} + \frac{1}{q}\{c_1 + c_2 + \ldots + c_q\} \ldots (\text{M}).$$

Supposing the errors in the comparisons to fluctuate evenly sometimes + and sometimes − and about the same amount on the average, we have

$$\frac{1}{q}\{c_1 + \ldots + c_q\} = 0,$$

and the larger q is the smaller will be the error, if any, arising from the probable assumption, and the equation is reduced to

$$m = \frac{m_1 + m_2 + \ldots + m_q}{q} + \frac{p}{q}\{\delta x_1 + \delta x_2 + \ldots + \delta x_q\} \ldots\ldots (\text{N}).$$

If before leaving A, and again after returning to A from B, the chronometers are rated in the usual way, we may obtain good probable values of δx_1 and δx_q.

Let a_1 and a_2 be the daily harbour rates of one of the chronometers rated at A immediately before and immediately after returning from B respectively, and β the harbour daily rate derived from the observations at B, I_1 and I_2 the number of days in the respective intervals between the observations at A before leaving for and after returning from B, and I the interval in days between the observations at B.

Then $\beta - a_1$ is the change in the daily rate of the chronometer in $\frac{I_1 + I_2}{2} + n_1$ days, and therefore the change in the daily rate on the outward voyage in one day $= \frac{2(\beta - a_1)}{I_1 + I + n_1}$, and on the return voyage the daily rate change will $= \frac{2(a_2 - \beta)}{I_2 + I + n_2}$.

From the middle of the outward voyage to that of the homeward voyage the number of days $= \frac{n_1 + n_2}{2} + I$, $\frac{n_1 + I}{2}$ on the outward, and $\frac{n_2 + I}{2}$ on the homeward, and therefore the difference in the average daily rate during the two voyages will be

$$\frac{(\beta - a_1)(n_1 + I)}{I_1 + I + n_1} + \frac{(a_2 - \beta)(n_2 + I)}{I_2 + I + n_2} = \delta x_1,$$

if the first chronometer be taken, and α_1, α_2, and β_3 are its harbour rates obtained in the manner before described.

Take the following example of four chronometers:

CHRONOMETERS FAST OF MEAN TIME AT A.

Date.	No. 1.	No. 2.	No. 3.	No. 4.
	h m s	h m s	h m s	h m s
July 18, - -	7 54 17·6	11 23 42·6	10 45 7·1	9 48 30
„ 23, - -	7 54 13·01	11 24 6·7	10 43 33·76	9 48 9·1
August 1, - -	7 54 5·72	11 24 50·3	10 40 46·1	9 47 33·7
„ 9, - -	7 54 0·65	11 25 28·15	10 38 17·9	9 47 4·1

CHRONOMETERS FAST OF MEAN TIME AT B.

Date.	No. 1.	No. 2.	No. 3.	No. 4.
	h m s	h m s	h m s	h m s
July 24, - -	7 56 47·15	11 26 46·3	10 45 50·2	9 50 40·15
„ 31, - -	7 56 41·72	11 27 20·82	10 43 40·5	9 50 13

For No. 1 chronometer:

At A from 18 to 23 July, lost $4^s{\cdot}59$, $\therefore I_1=5$ and $\alpha_1=-0^s{\cdot}918$;
„ 1 to 9 Aug., „ $5\cdot07$, $\therefore I_2=8$ „ $\alpha_2=-0\cdot634$;
At B from 24 to 31 July, „ $5\cdot43$, $\therefore I=7$ „ $\beta=-0\cdot776$;
and $n_1=n_2=1$.

$$\therefore \quad \delta x_1=\frac{0^s{\cdot}142\times 8}{13}+\frac{0^s{\cdot}142\times 8}{16}=0{\cdot}087+0^s{\cdot}071=0^s{\cdot}158.$$

For No. 2 chronometer:

At A from 18 to 23 July, gain $24^s{\cdot}1$, $\therefore I_1=5$ and $\alpha_1=+4^s{\cdot}82$;
„ 1 to 9 Aug., „ $37\cdot85$, $\therefore I_2=8$ „ $\alpha_2=\ 4\cdot73$;
At B from 24 to 31 July, „ $34\cdot52$, $\therefore I=7$ „ $\beta=\ 4\cdot93$;

$\therefore \quad \delta x_2=0^s{\cdot}11\times\frac{8}{13}-0{\cdot}2\times\frac{8}{16}=0^s{\cdot}068-0{\cdot}10=-0^s{\cdot}032.$

For No. 3 chronometer:

At A from 18 to 23 July, lost $93^s{\cdot}34$, $\therefore I_1=5$ and $\alpha_1=-18^s{\cdot}67$;
„ 1 to 9 Aug., „ $148\cdot2$, $\therefore I_2=8$ „ $\alpha_2=-18\cdot525$;
At B from 24 to 31 July, „ $129\cdot7$, $\therefore I=7$ „ $\beta=-18\cdot529$;

$\therefore \quad \delta x_3=8^s{\cdot}141\times\frac{8}{13}+0^s{\cdot}004\times\frac{8}{16}=0^s{\cdot}087+0^s{\cdot}002=0^s{\cdot}089.$

For No. 4 chronometer:

At A from 18 to 23 July, lost $20^s{\cdot}9$, $\therefore I_1=5$ and $\alpha_1=-4^s{\cdot}18$;
„ 1 to 9 Aug., „ $29\cdot6$, $\therefore I_2=8$ „ $\alpha_2=-3\cdot7$;
At B from 24 to 31 July, „ $27\cdot15$, $\therefore I=7$ „ $\beta=-3\cdot88$;

$\therefore \quad \delta x_4=0^s{\cdot}30\times\frac{8}{13}+0^s{\cdot}18\times\frac{8}{16}=0^s{\cdot}185+0^s{\cdot}09=0^s{\cdot}275.$

Therefore correction to arithmetic mean of the meridian distances

$$=\tfrac{1}{6}\{0^s{\cdot}158-0^s{\cdot}032+0^s{\cdot}089+0^s{\cdot}275\},$$
$$=\frac{0^s{\cdot}49}{6}=0^s.081.$$

$$\begin{array}{lr}
\text{Now} & m_1=2^m\ 35^s{\cdot}07 \\
& m_2=2\ \ 35\ {\cdot}06 \\
& m_3=2\ \ 35\ {\cdot}42 \\
& m_4=2\ \ 35\ {\cdot}175 \\
\hline
\text{Arithmetic mean, - -} & 2^m\ 35^s{\cdot}181 \\
\text{Correction, - - -} & +\ {\cdot}081 \\
\hline
\text{Meridian distance corrected,} & 2^m\ 35^s{\cdot}262.
\end{array}$$

Referring to the meridian distance before determined between Cape Passaro Castle and San Ranieri Tower, Messina, we see that the probable error of the mean of the observations made at Cape Passaro Castle on the 1st August is $\pm 0^s{\cdot}025$, and that the mean of those taken on the 9th August is $\pm 0^s{\cdot}022$; whilst the mean of the observations made at San Ranieri on the 2nd August gave a probable error of $\pm 0^s{\cdot}042$, and those on the 8th a probable error of $\pm 0^s{\cdot}02$. We therefore conclude that the probable errors made on the 1st, 8th, and 9th of August were all the same size, and half as large as those made on the 2nd August, and that if $\pm e$ be the error in the first three mentioned days, $\pm 2e$ will be that due to the observations made on the 2nd August.

The observations at San Ranieri made the standard chronometer lose $4^s{\cdot}5$ in six days, giving a harbour daily rate of $-0^s{\cdot}75$. The observations at Cape Passaro Castle made the same chronometer lose $5^s{\cdot}1$ in eight days—six in the harbour at San Ranieri and two at sea making the passages between the two places; and supposing the harbour rate given by the San Ranieri observations correct, the standard chronometer would have only lost $0^s{\cdot}6$ during the two passage days, and the daily passage rate would have been $-0^s{\cdot}3$. The weather being fine, and the temperature even during the nine days in question it is not at all probable that this large change in the rate of the chronometer really happened, especially as it was observed that the rates of all the other chronometers, which were going well together, were altered nearly to the same extent and in the same direction. We must therefore look somewhere else for the cause of this difference.

If we suppose that an error in the observations on the 1st August gave an error e in the mean time,

and those on the 9th August an error $-e$ in the mean time,

„ „ 2nd „ „ $-2e$ „ „

„ „ 8th „ „ $+e$ „ „

this will bring the harbour and passage rates more into harmony. Let also c be error arising from the comparisons of the observing watch with the standard chronometers, and having on the before named days the same sign as e, then we shall have on the 1st of August to apply a correction $e+c$ to the error of the standard chronometer, and on the 2nd August a correction $-(2e+c)$, on the 8th a correction $(e+c)$, and on the 9th $-(e+c)$.

Hence we have standard fast of Cape Passaro Castle mean time on August 1, -	$7^h\ 54^m\ 5^s{\cdot}7+e+c$,
Do. do. on August 9, - - -	$7\ \ 54\ \ 0{\cdot}6-e-c$,
Same chronometer fast mean time, San Ranieri, August 2, - - - -	$7\ \ 52\ \ 23{\cdot}7-2e-c$,
Do. do. on August 8, - - -	$7\ \ 52\ \ 19{\cdot}2+e+c$,

where e and c are expressed in seconds of time.

Let x also in seconds of time be the standard's harbour rate, and y its sea rate.

Then Cape Passaro observations give $6x+2y=-5^s{\cdot}1-2e-2c$.

From those at San Ranieri $6x=-4{\cdot}5+3e+2c$.

$$\therefore \quad 2y=-0^s{\cdot}6-5e-4c,$$

$$y=-0^s{\cdot}3-\frac{5e+4c}{2}, \text{ and } x=-0^s{\cdot}75+\frac{e}{2}+\frac{c}{3}.$$

Suppose we assume as possible that $e=c=0^s{\cdot}08$,

then $y=-0^s{\cdot}66$, and $x=-0^s{\cdot}75+0^s{\cdot}07=-0^s{\cdot}68$.

Hence half sum errors of standard on Cape Passaro Castle M.T. - - - -	$=7^h\ 54^m\ 3{\cdot}15$
Do. do. San Ranieri - -	$=7\ \ 52\ \ 21{\cdot}45-\frac{e}{2}$
	$=7\ \ 52\ \ 21{\cdot}41$
which gives San Ranieri Tower east of Cape Passaro Castle - - - -	$1\ \ 41{\cdot}74$

The advantage of running meridian distances in this manner over that usually adopted—viz., of rating the chronometers at one of the places, and running to the other as fast as possible and again rating the chronometers there—is so obvious that it will not be at all necessary to enlarge upon it. It is only a particular example of the general principle of not extending errors of observation, but to arrange every process in such a

manner that in the result the instrumental errors (regarding the observers as instruments) which must necessarily exist, and ought never to be lost sight of, shall have their effects diminished.

The following is an example of meridian distance run in a schooner of 100 tons on the coast of Nova Scotia, by the same person, using the same instrument, under favourable circumstances as to weather.

Year.	Month.	Day.	Place.	Chronometer Fast of Mean Time.	Half Sum.
				h m s	h m s
1861.	September.	4	Le Have, Nova Scotia,	5 35 26·17	5 35 35·755
		11	" "	35 45·34	
		5	Prospect, "	5 33 13·47	5 33 20·36
		10	" "	33 27·25	
		6	Halifax, "	5 32 34·66	5 32 38·68
		9	" "	32 42·7	

Taking the differences of the half sums in the foregoing table, we find

Halifax East of Prospect, - - $0^m\ 41^s{\cdot}68$
" " Le Have, - - 2 57·075

which are exactly the same as if we had determined the passage rates of the chronometer, and allowed for them in the usual way.

When the surveying staff is sufficient to occupy two stations on the shore at the same time and leave a sufficient number on board to wind up and compare the chronometers, and attend to the necessary duties on board, whilst the vessel is running backward and forward between the two stations, they can be taken at a greater distance from each other, because then the distance will be only limited by the distance the vessel can make certain of running in twenty-two hours. Two chronometers and two observers should be landed at each of the stations, which we will hereafter denote by A and B. The vessel, leaving two observers with the necessary instruments at A, proceeds to B, and lands the two other observers with the instruments. As soon as they are landed the observers set up their instruments and commence observations for time. The vessel at B, at a pre-arranged time, fires a given number of star rockets at fixed intervals, and the instant of their bursting is

taken by the chronometers on board, and by those on shore at *B*, and thus the chronometers are compared. Having ascertained that satisfactory observations for time have been made at *B*, the vessel immediately starts for *A* so as to compare her chronometers with *A*'s in the same way (exactly) at the same time on the following night, so that there may be as nearly as possible twenty-four hours between the two comparisons, and the same number of rockets fired at the same intervals, and as nearly as possible at the same hours of the night as those previously fired at *B*. After the comparisons at *A* have been completed, the ship returns to *B* so as to compare chronometers there, exactly in the same way, and at the same hour on the following night. She then returns again to *A*, and after the comparisons have been made there at the same hour on the next night, and the observers at *A* have completed their observations for time, the first series of observations will be completed. The *A* observers should, with their instruments, be transferred to *B*, and the *B* observers brought to *A* with their instruments, and another series of exactly similar observations made. By this means the personal and instrumental errors, more especially the former, which in some instances I have found to amount to $0^s{\cdot}6$, and therefore not to be disregarded, are eliminated.

The following example of comparing chronometers in the way before recommended by observing the times of the bursting of star rockets will show with what degree of accuracy this method can attain.

The *Hydra*'s standard chronometer on board was compared with chronometer *C* at Gozo Lighthouse at 9^h P.M. on the 20th August 1867, by taking the times at which star rockets, fired from the ship at given times, burst.

Distinguishing Number of Rocket.	Time of the Rockets Bursting.		C. Fast of S.
	Chronometer S. on Board.	Chronometer C. on Shore.	
	h m s	h m s	h m s
1	4 52 30·2	6 29 46·2	1 37 16·0
2	57 19·3	34 35	15·7
3	59 8·8	36 24·2	15·4
4	5 1 10·9	38 26·4	15·5
5	3 10·2	40 26·2	16·0
6	8 17	45 33	16·0

Arithmetic mean, - - - - - $1^h\ 37^m\ 15^s{\cdot}77$

The probable error in one of these comparisons is $\pm 0^s{\cdot}23$, and that of the arithmetic mean $\pm 0^s{\cdot}84$.

Similar comparisons were made at the same place, by the same persons, exactly in the same way, on the 23rd August 1867, and gave the following results :

Rocket Number.	Chronometer Time of Bursting.		C. Fast of S.
	S. on Board.	C. on Shore.	
	h m s	h m s	h m s
1	4 48 5·9	6 25 21·1	1 37 15·2
2	55 1·1	32 15·9	14·8
3	57 2·6	34 17·5	14·9
4	59 4·4	36 19·5	15·1
5	5 1 2·4	38 17·1	14·7
6	4 2·9	41 17·7	14·8
7	7 58·9	45 13·9	15·0

Arithmetic mean, - - - - - $1^h\ 37^m\ 14^s{\cdot}93$

The probable error in the comparison given by one rocket's bursting is $\pm 0^s{\cdot}05$, and that of the arithmetic mean of the whole $7 \pm 0^s{\cdot}02$.

The errors of the chronometers on shore can be determined by observing equal altitudes of the sun with a sextant and artificial horizon in the manner already described, or by observing with the same instruments equal altitudes of stars at night. When this means is adopted the same star should be observed at both places at the same altitudes, and the rockets fired, so that the mean of the times of their bursting should be as nearly as possible the time of the star's transit across the meridian. Take for example the following observation on the star α Pegasi at observing station Δ in latitude 43° 33′ 17″ N. and longitude $4^h\ 21^m\ 38^s$ W. made on the 1st September 1858. The double altitudes of the star which were observed were selected before landing.

2 × Alt. α Pegasi.	Sidereal Chronometer Time.		Sums.
	A. Meridian.	P. Meridian.	
°	h m s	h m s	h m s
60	4 32 7·5	12 49 6·3	17 21 13·8
61	34 35·2	46 19	14·2
62	37 42·4	43 31	13·4
63	40 31	40 42·7	13·7
64	43 19·7	37 54·5	14·2
65	46 8·2	35 6	14·2
66	48 57·4	32 16·4	13·8
67	51 47·6	29 26	13·6
68	54 37·5	26 36·6	14·1
69	57 28	23 46·5	14·5
70	5 0 17·8	20 56·2	14·0
71	3 8·5	18 5·4	13·9

Arithmetic mean, - - - - -	$17^h\ 21^m\ 13^s{\cdot}95$
Half sum, - - - - - -	$8^h\ 40^m\ 36^s{\cdot}97$
Right ascension of α Pegasi, - - -	22 57 44·98
Chronometer fast of sidereal time, - -	$9^h\ 42^m\ 52^s$

Probable error of one observation $\pm 0^s{\cdot}125$
" " arithmetic mean $\pm 0{\cdot}01$

Instructions to the surveyor in charge of station *A*, 1 Sept. 1858:

"Observe equal altitudes of α Pegasi, commencing at sextant reading 60°, putting the index bar on 1° *exactly* after each coincidence has been observed, as far as sextant reading 71° inclusive, being very careful to set the index bar *exactly* at the degree division, and to clamp it firmly. The sidereal chronometer time of the first observation will be about $4^h\ 32^m$, and the time between two consecutive coincidences about $2^m\ 50^s$. Time is the most important object of this night's observations; but if, without tiring the eye, circum-meridian altitudes can be taken for latitude, take six coincidences of α Aquarii east of the meridian, and six to the west. It will pass south of the zenith about 44° 34′. The time of this star's meridian passage by the sidereal chronometer is estimated to be $7^h\ 41^m$ nearly. Afterwards take twelve coincidences of Polaris—six before $10^h\ 50^m$ by the sidereal chronometer, and six at about equal intervals after. The altitude in the mercurial horizon will be about 90°.

"Rockets will be fired from the ship bearing S. 62° E. or thereabout from *A* station, at as nearly as possible the following times by the observing chronometer at *A*.

1st rocket to be fired at			- - -	8^h 32^m
2nd	"	"	- - -	8 35
3rd	"	"	- - -	8 38
4th	"	"	- - -	8 41
5th	"	"	- - -	8 44
6th	"	"	- - -	8 47
7th	"	"	- - -	8 50

"Look out for the same number of rockets to be fired from the same place, at the same hours and minutes, on the 4th September next. In the meantime take the same observations on the same stars every night."

The same orders were given to the surveyor in charge at B, except that the rockets at that place were fired on the 2nd and 3rd September respectively.

I may here observe that it was a standing order to all the surveyors to determine the index error of the sextant by taking coincidences of the direct and reflected images of the same star whenever observations for latitude were made.

A four-foot portable transit judiciously used will determine the errors of the chronometers more conveniently and accurately than observations made with a sextant. Two such instruments are required to run the meridian distance between two places, one to be set up at each, and the transit of the *same* stars across their respective meridians observed at both during the same nights. Each transit should be provided with a small portable observatory to protect it from the weather—a very strong portable stand on which to secure the stand of the portable transit, and a detached level to level its top. It is very important to have good stride levels to level the axis of the transit, and thereafter to determine its level error accurately. The stars to be observed should be selected by the director of the survey from the British Association Catalogue, and tabulated for each observatory with the magnitude, approximate right ascension declinations, and zenith distances of the stars.

A complete set of observations consists of

1. A star passing very near the zenith.
2. Two circum-polar stars passing below the pole.
3. Two zenith stars passing near to but on the opposite side of the zenith from that of the elevated pole.
4. Two equatorial stars very near to the equator, but on the opposite side to that of the zenith.

The chronometers are carried in the ship backward and forward between the observatories, and compared with the respective observatory chronometers, exactly in the same manner as before described.

The transit instrument should be placed at the centre of a main station, or as near thereto as possible, and so as to have a good clear view either north or south, so that a meridian mark may be placed in a convenient position. The approximate direction of the meridian can be determined by observations on the sun made with a theodolite; and, having found the theodolite reading of the north or south line, as the case may be, set the vernier of its horizontal plate at the reading thus ascertained, clamp it, and set up the meridian mark, so that the cross of the telescope wires may intersect it. A suitable mark (Fig. 10) consists of a straight vertical pole ACB

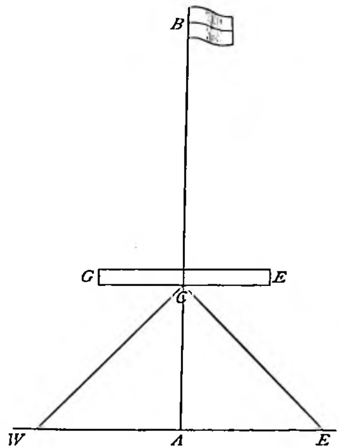


Fig. 10

about twenty feet or more long, fixed in the plane of the meridian passing through the transit instrument's place by four strong shores hinged at C about ten feet from A, the lower end of the pole. CE and CW, the east and west shores, must be especially strong and firm, so as to keep the point C rigidly fixed in the plane of the meridian, or as near thereto as possible. GF is a horizontal shelf, about four feet long, standing east and west. The bottom of the shelf is graduated from

a zero mark at the point C in inches and tenths towards F and G respectively. On this shelf the light to be seen at night is placed, at first on the zero mark at C; afterwards it is moved towards F or G, when, by a night's observation with the transit, the correct direction of the meridian that should be traced out by its line of collimation has been determined. A red and white flag placed at B, the top of the meridian pole, will serve to distinguish it during the day. This mark is convenient, but not essential; otherwise a polar star must be used, the sidereal time being known within a few seconds. The middle wire of the transit, after its axis has been carefully levelled, is made to intersect the star at the estimated time of its meridian passage, as shown by the sidereal chronometer. This will bring its middle wire sufficiently near the meridian to commence the series of observations before recommended.

The transit instrument being on its Y's, with the illuminated end of its axis east, well levelled, the reading of the level carefully read and noted, and the middle wire brought into the meridian by means of the deviation screws, using either the light on the meridian mark or a star very near the pole passing the meridian, set the altitude circle to the zenith distance of the first selected zenith star, turn the telescope round on its axis until the bubble of the setting circle spirit-level comes to its central position. The probable time of the star entering the field of view of the transit telescope will be pointed out by the sidereal chronometers, the error of which on sidereal time is approximately known when the star enters the field of view. If not near the horizontal diameter, it must be brought there as nearly as possible, by slightly turning the telescope round its transverse axis. Shortly before the star reaches the first wire, the assistant must be directed to count the seconds of the sidereal chronometer. The time of the star passing the first wire is estimated by the observer, and noted in the observation book by his assistant. In a similar manner its passage across the other six transit wires must be observed. The arithmetic mean of the seven times thus recorded will give the time shown by the sidereal chronometer at the instant of the star's passage across the mean wire of the transit. The level readings must then be carefully taken and noted; the level error can now be calculated and applied; the errors of collimation and deviation are not known, but the first ought previously to have been made very small by means of the adjusting screws of the diaphragm carrying the transit wires, reversing the telescope on a distant well-defined object, and moving the diaphragm until the middle wire of the transit exactly coincides with it in both positions. The latter will have little or no apparent effect

on stars passing very near to the zenith. Comparing the time of the passage of the star across the meridian, observed as above described, with the star's right ascension, we have a near approximation to the error of the sidereal chronometer on the sidereal time of the place of observation; and hence the chronometer time at which the first selected circum-polar star will cross the first transit wire becomes very nearly known. The transit telescope is then pointed to catch the star as it enters the field of view. The time of its crossing the first wire is noted, and if sufficiently near the estimated time to ensure the deviation being small, as will be hereafter explained, the times of the star crossing the second, third, and middle wires respectively are then observed and noted, and when the observation on the middle wire is completed, the transit telescope must be carefully lifted off its Y's and reversed, so as to have the illuminated end of its axis west. The time of the star's transit across the third, second, and first wires are then observed and noted; the level must then be carefully observed and noted. The sum of the chronometer times of the star's transit across the three wires in their two positions, corrected for level error, will give the time of its transit across the line of collimation of the transit had its axis been level on both occasions.

The other circum-polar star is then observed in the same way, only the seventh, sixth, fifth, and middle wires are observed instead of the first, second, third, and middle wires, and the time of the star's transit across the line of collimation of the telescope with its axis level obtained.

The level *must* always be carefully observed before and after each observation, and when not mentioned hereafter will always be supposed to have been done.

Next observe two stars passing near to, but on the equatorial side of the zenith, the first with the illuminated end of the axis east over *all* the seven wires. The transit telescope must be carefully reversed, so as to have the illuminated end of its axis west, and the second zenith star observed over all the wires.

Lastly, two equatorial stars must be observed, the first with the illuminated end of the axis of the transit telescope west over all the wires; and then, the transit telescope having been carefully reversed so as to have the illuminated end of its axis east, the second must be observed across all the wires.

From these observations the instrumental errors, and that of the sidereal chronometer on sidereal time, can be determined with considerable accuracy.

The portable observatory should be erected so that the

middle of the aperture through which the stars are observed may coincide with the centre of the station, and the sides of the aperture be parallel to the approximate direction of the meridian previously ascertained. The portable stand is then placed firmly in the ground, with its top well levelled, for which a detached portable level must be used. The top must be so firm and secure as to be incapable of twisting, and not liable to get out of level. Very great attention must be paid to this. When the stand is securely fixed in its proper position the iron stand of the portable transit telescope must be firmly screwed to it in such a position that the adjusting screws of its moveable Y being so arranged that it is in its central position, both as regards level and deviation, the straight line joining the bottoms of the Y's may be as nearly horizontal and east and west as possible.

The reasons for arranging the observations in the way before recommended may be shortly stated as follows:

By carefully levelling the transverse axis of the transit telescope, the level error becomes small. By carefully reading the level in both its positions, both before and after each observation of a star's passage across the meridian, this small level error becomes known. This being done, the line of collimation of the transit telescope will describe a circle very nearly coinciding with a vertical circle passing through the zenith. The error of collimation is only known to be very small if the proper means have been taken, *which they ought*, to make it so. The deviation is unknown, and is probably large in comparison with the other two errors.

A zenith star is therefore selected in the first instance, because the deviation will have a smaller effect on its time of transit than that of stars passing farther from the zenith, and thus the error of the chronometer on sidereal time will be more accurate when obtained from a zenith star than from others which are far from the zenith. Two circum-polar stars passing the meridian below the pole are next observed, because the deviation of the transit telescope will have a much larger effect on the times of their transit across the meridian than on those of other stars, and by reversing the transit telescope, as before described, we obtain data to determine the collimation errors of all the wires.

Two equatorial stars are selected, because they cross the transit wires more rapidly than others nearer the poles, and can therefore be observed with greater precision; and by making the two observations with the telescope in reversed positions, they will be oppositely affected by collimation error—so that any error in the determination of the collimation error

will not have any sensible effect on the mean of the two errors of the chronometer obtained from the observations.

We will now examine the effects which the errors in the position of the transit telescope produce on the time of a star's passage across its mean wire.

Let $MZPN$ (Figs. 11, 12, and 13) be the meridian of the place

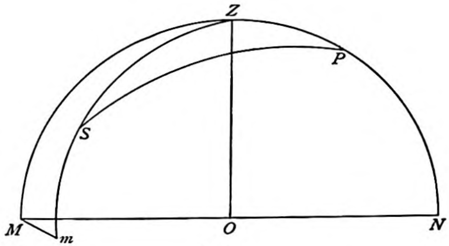


FIG. 11.

Z the zenith, and P the pole. Referring to Fig. 11, let ZSm be the arc of a great circle passing through Z, traced out by the line of collimation of the transit telescope revolving about its axis O. The angle MZm is the deviation of the plane of collimation of the transit from the plane of the meridian. We will denote this angle by D.

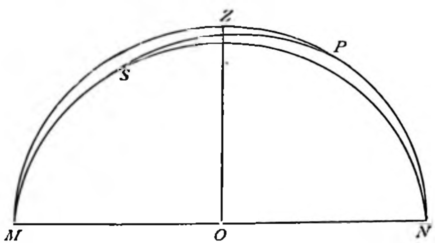


FIG. 12.

If the transverse axis of the telescope of the transit is not exactly horizontal, the great circle traced out by its line of collimation will take a position like MSN (Fig. 12), which does not pass through Z. This is called the error of level, and is measured by the angle $ZMS = L$ the level reading.

If the mean of the wires does not coincide with the line of collimation of the transit telescope, the mean of the wires will

trace out a circle mSn (Fig. 13) parallel to the great circle $MZPS$, and if Ss be the arc of a great circle passing through S and cutting MsZ at right angles, then $Ss=C$ is the error of collimation of the transit telescope.

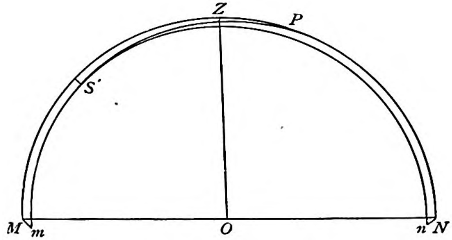


FIG. 13.

In Figs. 11, 12, and 13, let $ZS=z$, $PS=\Delta$, and $ZP=c$.

Referring to Fig. 11, let $ZPS=d$; then d is the error in the time of S's transit across the meridian, in consequence of the error D of deviation.

In the triangle ZPS

$$\sin d \,.\, \sin \Delta = \sin D \,.\, \sin (\Delta - c) \text{.................} (1)$$

D and d are generally so small that they may be taken for their sines without sensible error,

and
$$d=D\frac{\sin (\Delta - c)}{\sin \Delta} \text{..........................} (2)$$

Referring to Fig. 12, and putting $MPS=l$ the error in the S's transit consequent on the error L of level, we have, since l and L are both very small,

$$l=L\frac{\text{cosin } (\Delta - c)}{\sin \Delta} \text{..........................} (3)$$

Lastly, referring to Fig. 13, and putting $MPS=c$ the error in the time of S's transit arising from the error C of collimation, we have from the right-angled triangle PSs, right angled at S,

$$\sin c \,.\, \sin \Delta = \sin C \text{.......................} (4)$$

or since c and C are in general both small, unless Δ is very small, we may take without sensible error

$$c = C \text{ cosec } \Delta \text{..........................} (5)$$

From equation (2) we see that d varies with $\frac{\sin \overline{\Delta - c}}{\sin \Delta}$, and is therefore least when $\Delta = c$, in which case it vanishes altogether, except for a star exactly at the pole.

Equation (3) shows that l varies with $\frac{\text{cosine}\,(\Delta-c)}{\sin\Delta}$, and has therefore its largest value when $\Delta-c=0$; that is for a star passing through the zenith.

From equation (5) we see that c varies with the cosecant of the star's polar distance, and is therefore least when $\Delta=90°$ or for equatorial stars, and greatest for polar stars.

Let $C_1, C_2, \ldots C_7$ be the respective collimation errors of the seven vertical wires of the transit telescope, reckoning from the illuminated end of the axis; also let $C_e=\frac{C_1+C_2+C_3}{7}$ and $C_w=\frac{C_5+C_6+C_7}{7}$; then $C=\frac{C_1+C_2+C_3+C_4+C_5+C_6+C_7}{7}$.

When Δ is very small equation (4) must be used, so that for circumpolar stars we must take $\sin C=\sin c\,.\sin\Delta$. If therefore t_1 be the time by the clock when a circumpolar star crosses the first wire of the transit, T_1 the time at which it crosses the line of collimation, and t_1' the time at which it again crosses the first wire after the telescope axis has been reversed, so as to have its illuminated end on the opposite side to that which it had when the first observation was taken—

In the first position of the telescope $c_1=T_1-t_1$,

„ second „ „ $c_1=t_1'-T_1$.

Hence $2c_1=t_1'-t_1$

and $T_1=\frac{t_1'+t_1}{2}$.

Substituting in equation (4) the value of c_1 thus determined, we have

$$\sin C_1=\sin\frac{t_1'-t_1}{2}\sin\Delta\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

from which C_1 is easily calculated. After the passage of the star over the fourth wire the telescope is reversed; and observations being made over the second and third wires similar to those we have just described over the first wire, we shall have in like manner

$$\sin C_2=\sin\frac{t_2'-t}{2}\sin\Delta,$$

$$\sin C_3=\sin\frac{t_3'-t_3}{2}\sin\Delta,$$

from which C_2 and C_3 will be determined.

From the second circumpolar star the values of C_5, C_6, and C_7

will be found in the same way. From the observations of the passages of the stars across the first three wires, we have

$$T_1=\frac{t_1+t_1'}{2};\ T_1=\frac{t_2+t_2'}{2};\ T_1=\frac{t_3+t_3'}{2}.$$

Taking the arithmetic mean of these three equally good values of T_1, we find that

$$T_1=\frac{t_1+t_2+t_3+t_1'+t_2'+t_3'}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

From this value of T_1 we obtain that of c_4, since

$$c_4=T_1-t_4,$$

$$\therefore \quad \sin C_4=\sin(T_1-t_4)\sin\Delta;$$

or, since T_1-t_4 is generally pretty small,

$$C_4=(T_1-t_4)\sin\Delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

will give the value of C_4.

From the second circumpolar star we get another equally good value of C_4, and also the values of C_5, C_6, and C_7. Combining these the value of

$$C_,=\frac{C_1+C_2+C_3+C_4+C_5+C_6+C_7}{7} \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

is immediately found.

From the observations on the second polar star we find T_2 the time of its transit across the line of collimation of the transit telescope in the same way as T_1 was determined by equation (7).

Let D be the deviation of the line of collimation from the meridian, considered positive when to the east of the elevated pole, L the level error of the axis of the telescope when the first circumpolar star was observed, considered positive when the east end of the axis is the higher; also let

$$\alpha_1=\cos(\Delta_1-c)\operatorname{cosec}\Delta_1,$$
$$\beta_1=\sin(\Delta_1-c)\operatorname{cosec}\Delta_1,$$
$$\gamma_1=\operatorname{cosec}\Delta_1,$$

where Δ_1 is the polar distance of the first observed circumpolar star, and the clock time of the passages may be expressed as follows:—

First circumpolar star $=T_1-\alpha_1L_1+\beta_1D$,
Second „ „ $=T_2-\alpha_2L_2+\beta_2D$.

Let E be the error of the sidereal clock or chronometer, as the case may be, at the middle time between the two circumpolar stars and the two zenith stars; the rate, being small and

unknown, may be left out of consideration as not having a sensible effect on D, which it is our object here to determine.

Hence neglecting the clock rate, and putting A_1 for the right ascension of the first circumpolar, and A_2 for that of the second, we have

$$A_1+E=T_1-a_1L_1+\beta_1D \text{ very approximately..........(10)}$$
$$A_2+E=T_2-a_2L_2+\beta_2D \quad \text{,,} \quad \text{,,} \quad \text{..........(11)}$$

In a similar manner from the two zenith stars we have

$$A_3+E=T_3-a_3L_3+\beta_3D+\gamma_3C \text{ very approximately....(12)}$$
$$A_4+E=T_4-a_4L_4+\beta_4D-\gamma_4C \quad \text{,,} \quad \text{,,} \quad (13)$$

Adding and transposing,

$$2E=T_3-A_3+T_4-A_4-(a_3L_3+a_4L_4)+(\beta_3+\beta_4)D+(\gamma_3-\gamma_4)C;$$

adding (10) and (11),

$$2E=T_1-A_1+T_2-A_2-(a_1L_1+a_2L_2)+(\beta_1+\beta_2)D.$$

Eliminating E we have

$$D=\frac{T_3-A_3+T_4-A_4-T_1+A_1-T_2+A_2+a_1L_1+a_2L_2-a_3L_3-a_4L_4+(\gamma_3-\gamma_4)C}{\beta_1+\beta_2-\beta_3-\beta_4}\text{....(14)}$$

very approximately.

Let r be the rate of the timepiece in seconds per hour, and let A_1, A_2, A_3, A_4 be expressed in hours, then T_3-T_1 will be $(A_3-A_1)r$ larger than if the timepiece went exactly to sidereal time, and in the same way T_4-T_2 will be $(A_4-A_1)r$ larger than on the supposition that $r=0$; and therefore referring to equation (14), we see that D will be larger than it ought to be by $\frac{(A_3-A_1+A_4-A_2)}{\beta_1-\beta_3+\beta_2-\beta_4}r$, a very small quantity, since we suppose the timepiece to have a very small rate, and β_3 and β_4 are respectively very small in proportion to β_1 and β_2; besides the numerator $A_3+A_4-A_1-A_2$ cannot be large, and therefore we may suppose equation (14) will give D with sufficient accuracy for our present purpose. From the values of C and D thus found, the observations on the zenith and equatorial stars are corrected, and the arithmetic mean of the four errors of the clock thus found will give its error E in sidereal time at the middle time between the observations, or at sidereal time $\frac{A_1+A_2+A_3+A_4}{4}$; that is, by regarding the mean of the two zenith stars as one complete observation, and the mean of the two equatorial stars as the other, in this way the clock rate will be eliminated, because it equally affects each pair of observations, but in opposite directions. The clock rate will be

determined by observations made on succeeding days, and the value of *D* corrected if necessary, as well as the observations from which *E* was found.

The following example will serve to illustrate the preceding.

On the 1st December 1864, in latitude 30° 20′ N, a good four-feet portable transit was got approximately into the meridian and levelled; the adjusting screws of the Y carrying the transverse axis of the telescope were purposely kept in their mean positions; the instrument was furnished with a good stride level, and the transit wires were carefully adjusted so as to make the collimation error of the middle very small; the sidereal chronometer was known to be approximately about $2^h\ 7^m\ 20^s$ fast on sidereal time with a small daily rate.

I was determined to use Polaris to bring the line of collimation of the telescope as near as possible into the meridian, so as to reduce the deviation as much as possible before commencing the observations. The telescope was placed with the illuminated end of its axis east, the equatorial distance of the 7th vertical wire from the line of collimation was about 48′, and that of the 6th wire 32′; the times at which Polaris ought to cross the 6th and 7th wires respectively were calculated as follows, the polar distance of Polaris being at that time 1° 24′ 20″·5:

For 6th wire log sin 32′, - - -	7·96887	For 7th wire log sin 48″, - - -	8·14495
For log sin (1° 24′ 20″·5), - -	8·38972	- - - -	8·38972
Log sin ($1^h\ 29^m\ 12^s$),	9·57915	Log sin ($2^h\ 18^m\ 46^s$),	9·75523

Polaris' right ascension, - - -	1^h	10^m	32^s
Interval from meridian to 6th wire, -	1	29	12
Time of Polaris' crossing 6th wire, -	2^h	39^m	44^s
Sidereal chronometer fast of S.T., - -	2	7	20
Chronometer time nearly at which Polaris ought to cross the 6th wire, - -	4^h	47^m	4^s
Interval from meridian to 7th wire, -	2^h	18^m	46^s
Sidereal time of Polaris' crossing 7th wire,	3^h	29^m	18^s
Time by sidereal chronometer, - -	5	36	38

In the same way the chronometer time at which Polaris

ought to be exactly equidistant between the 6th and 7th wires was found to be $5^h\ 11^m\ 8^s$.

At chronometer time 5^h, the telescope was set at Polaris' zenith distance, and upon looking through the telescope Polaris was seen between the 6th and 7th wires, and by means of the Y's deviation screws, the axis was turned in azimuth, so that at $5^h\ 11^m$ it appeared to be exactly midway between the 6th and 7th wires; the deviation screws were tightened up carefully, and the axis which had been thrown a little out was carefully relevelled, and the level readings noted.

The time of Polaris' passage across the 7th wire was observed, and the chronometer time $5^h\ 35^m\ 59^s$ noted; comparing this with the estimated time we see that it is 39^s sooner than it should have been; but 1^s deviation in the line of collimation of the transit affects Polaris' time of transit by about 30^s, and therefore we conclude that the deviation has been made small, and the errors of level and collimation being also small, the instrument was considered in a good condition for making the following observations which had been pre-arranged.

First circumpolar observed was ζ Ursæ Minoris.

Right ascension $15^h\ 48^m\ 49^s{\cdot}05$, declination 78° 12′ 30″ N, crossing the meridian below the pole; the illuminated end of the axis of the transit telescope was east, the sitting circle adjusted to the star's altitude, the telescope pointed to the altitude at which the star would cross the meridian, and the level in both positions read off and noted; the passage of ζ Ursæ Minoris across the 7th, 6th, 5th, and 4th wires were then observed and noted in the observatory book, and gave as follows:—

Chronometer time of passage across	7th wire, -	5^h	40^m	$33^s{\cdot}1$
„ „ „	6th „ -		45	47·4
„ „ „	5th „ -		51	0·7
„ „	middle or 4th „ -		56	13·8.

Level read, and telescope reversed with illuminated end of the axis west.

Chronometer's time of passage across	5th wire,	6^h	1^m	$27^s{\cdot}3$
„ „ „	6th „		6	40·5
„ „ „	7th „		11	54·9.

Level read—mean of all the level readings gave the east end of the axis $0^s{\cdot}13$ too high.

The transit was then prepared for observing ϵ Ursæ Minoris about to cross the meridian below the pole; its right ascension was $16^h\ 59^m\ 43^s{\cdot}5$, and declination 82° 15′ 20″ N; level carefully read and noted.

ϵ Ursæ Minoris crossed 1st wire at chron. time,			6^h	43^m	23^s
,,	,,	2nd ,,		51	19·1
,,	,,	3rd ,,		59	14·9
,,	,,	4th or middle wire ,,	7	7	9·9.

Level read, and telescope reversed with illuminated end of its axis east.

Chronometer time of passage across 3rd wire,			-	7^h	15^m	4^s·5
,,	,,	,, 2nd ,,	-		23	0·2
,,	,,	,, 1st ,,	-		30	56·4.

Level read—mean of all its readings gave the east end of its axis 0^s·11 too high.

Next came two equatorial stars, viz.:

ϵ Orionis, R.A.,	5^h	29^m	23^s	declination,	1°	17′	30″ S,
and α Orionis, ,,	5	47	53·49	,,	7	22	40 N.

Illuminated end of axis east. Level read and noted.

ϵ Orionis crossed 1st wire at chronometer time				7^h	33^m	33^s·6
,,	,,	2nd	,, ,,		34	37·4
,,	,,	3rd	,, ,,		35	41·6
,,	,,	4th	,, ,,		36	45·5
,,	,,	5th	,, ,,		37	49·6
,,	,,	6th	,, ,,		38	53·5
,,	,,	7th	,, ,,		39	57·6.

Level read, giving (mean of all) east end of axis 0^s·07 too high. Telescope was then reversed, and the level read and noted.

α Orionis crossed 7th wire at chronometer time				7^h	52^m	1^s·9
,,	,,	6th	,, ,,		53	7·5
,,	,,	5th	,, ,,		54	11
,,	,,	4th	,, ,,		55	15·5
,,	,,	3rd	,, ,,		56	20
,,	,,	2nd	,, ,,		57	24·6
,,	,,	1st	,, ,,		58	29.

Level read and noted; mean of level readings gave east end of axis 0^s·1 too high.

The telescope was reversed with illuminated end of axis east to observe a pair of zenith stars:

α Geminorum R.A.,	7^h	26^m	0^s·6,	declination,	32°	10′	40″ N.
6 Cancri ,,	7	55	14·77	,,	28	10	N.

Level readings before and after α Geminorum was observed gave east end of axis 0^s·12 too high.

α Geminorum crossed the 1st wire at chron. time	9^h	29^m	$35^s{\cdot}6$
„ „ 2nd „ „		30	51
„ „ 3rd „ „		32	6
„ „ 4th „ „		33	22 ·3
„ „ 5th „ „		34	38
„ „ 6th „ „		35	53 ·4
„ „ 7th „ „		37	8 ·9.

Telescope reversed with illuminated end of its axis west, and pointed to 6 Cancri. Level read:

6 Cancri passed the 7th wire at chron. time	9^h	58^m	$58^s{\cdot}8$
„ „ 6th „ „	10	0	11 ·5
„ „ 5th „ „		1	24 ·1
„ „ 4th „ „		2	36 ·6
„ „ 3rd „ „		3	49 ·2
„ „ 2nd „ „		5	1 ·7
„ „ 1st „ „		6	14 ·3.

Level read, mean of all the readings gave the east end of the axis $0^s{\cdot}15$ too high.

First taking the observations on ζ Ursæ Minoris:

Illuminated end of axis east, it crossed 7th wire at chronometer time,				5^h	40^m	$33^s{\cdot}1$
Do.	do.	do.	6th		45	47 ·1
Do.	do.	do.	5th		51	0 ·7
Do.	do.	west	5th	6	1	27· 3
Do.	do.	do.	6th		6	40· 5
Do.	do.	do.	7th		11	54· 9

T_1 = arithmetic mean of above - - - = $5^h\ 56^m\ 13^s{\cdot}98$

Referring to equation (6) $\sin C_7 = \sin \frac{t_7' - t_7}{2} \sin \Delta$,

$= \sin (15^m\ 40^s{\cdot}9) \sin (11°\ 47'\ 30'')$,

log sin ($15\ \ 40^s{\cdot}9$), - - -	8·834871
„ ($11°\ 47'\ 30''$), - - -	9·310382
log sin $C_7(=0°\ 48'\ 2'')$, - -	8·145253

In like manner it will be found that

$C_6 = 32'\ 0''$, and $C_5 = 16'\ 0''{\cdot}4$, and $C_4 = (T_1 - t_4) \sin \Delta$,
$= 0{\cdot}18 \sin (11°\ 47'\ 30'')$,
$= 0^s{\cdot}0365$.

In like manner from the observations on ϵ Ursæ Minoris, we have
$T_2 = 7^h\,7^m\,9^s{\cdot}68$; $C_1 = 47'\ 58''{\cdot}8$, $C_2 = 31'\ 59''{\cdot}8$, $C_3 = 15'\ 59''{\cdot}5$, and $C_4 = 0^s{\cdot}0296$. Taking the arithmetic mean of this and the former value of C_4 we find $C_4 = 0^s{\cdot}033$.

These values of C are taken irrespective of sign; but supposing when the illuminated end of the axis is east we take C_1, C_2, and C_3, as positive, and C_5, C_6, and C_7 as negative, then C_4 having the same sign as C_5, C_6, and C_7 will be negative also; putting these values in time,

$$\begin{array}{lrr}
C_1 = & +3^m & 11^s{\cdot}92 \\
C_2 = & 2 & 7\ {\cdot}99 \\
C_3 = & 1 & 3\ {\cdot}97 \\
C_4 = & -0 & 0\ {\cdot}03 \\
C_5 = & -1 & 4\ {\cdot}03 \\
C_6 = & -2 & 8\ {\cdot}0 \\
C_7 = & -3 & 12\ {\cdot}13 \\
\hline
C\ = & -0 & 0\ {\cdot}044
\end{array}$$

, when the illuminated end of the axis is east, and $+0^s{\cdot}044$ when it is west.

Next, the two zenith stars α Geminorum and 6 Cancri.

First referring to the notation in equations (10), (11), (12), (13), and (14), and taking α Geminorum to represent the third star, we have $A_3 = 7^h\,26^m\,0^s{\cdot}5$, $T_3 = 9^h\,33^m\,22^s{\cdot}27$, $\therefore T_3 - A_3 = 2^h\,7^m\,21^s{\cdot}77$, $\Delta_3 = 57°\ 39'\ 40''$, $\therefore \Delta_3 - C = -(1°\ 50'\ 40'')$, $a_3 = 1{\cdot}181$, $\beta_3 = -0{\cdot}039$, $\gamma_3 = 1{\cdot}182$, $C = -0^s{\cdot}044$, $L_3 = 0^s{\cdot}12$.

Taking 6 Cancri as the fourth star with the same notation, we have $A_4 = 7^h\,55^m\,14^s{\cdot}77$, $T_4 = 10^h\,2^m\,36^s{\cdot}6$, $\therefore T_4 - A_4 = 2^h\,7^m\,21^s{\cdot}83$, $\Delta_4 = 61°\ 50'$, $\therefore D_4 - C = 2°\ 10'$; $L_4 = 0^s{\cdot}15$, $a_4 = 1{\cdot}133$; $\beta_4 = 0{\cdot}043$, and $\gamma_4 = 1{\cdot}134$.

From the two circumpolar stars, for the first ζ Ursæ Minoris with the same notation, we have $A_1 = 3^h\ 48^m\ 49^s{\cdot}05$; $T_1 = 5^h\ 56^m\ 13^s{\cdot}98$, and $\therefore T_1 - A_1 = 2^h\ 7^m\ 24^s{\cdot}93$, $a_1 = 1{\cdot}56$, and $\beta_1 = 4{\cdot}64$.

The second circumpolar star, ϵ Ursæ Minoris, gives $A_2 = 4^h\ 59^m\ 43^s{\cdot}5$, $T_2 = 7^h\ 7^m\ 9\ {\cdot}68$, $\therefore T_2 - A_2 = 2^h\ 7^m\ 26^s{\cdot}18$, $a_2 = 2{\cdot}85$, $\beta_2 = 6{\cdot}85$, observing that A_1 and A_2 represent the sidereal times respectively at which the stars cross the meridian *below* the pole, and cause the sign of L and D to change, and therefore $L_1 = -0^s{\cdot}13$ and $L_2 = -0^s{\cdot}11$.

Referring to (14) we see that

$$\begin{aligned}
&T_3 - A_3 + T_4 - A_4 + A_1 - T_1 + A_2 - T_2 \\
&\quad = 2^h\,7^m\,21^s{\cdot}77 + 2^h\,7^m\,21^s{\cdot}83 - 2^h\,7^m\,24^s{\cdot}93 - 2^h\,7^m\,26^s{\cdot}18 \\
&\quad = -7^s{\cdot}51\,;
\end{aligned}$$

also $a_1L_1 = -0^s{\cdot}203$, $a_2L_2 = -0^s{\cdot}313$, $a_3L_3 = 0^s{\cdot}142$, $a_4L_4 = 0^s{\cdot}17$,

$$\therefore a_1L_1 + a_2L_2 - a_3L_3 - a_4L_4 = -0^s{\cdot}203 - 0^s{\cdot}313 - 0^s{\cdot}142 - 0^s{\cdot}17$$
$$= -0^s{\cdot}828,$$
$$\gamma_3 - \gamma_4 = 1{\cdot}18 - 1{\cdot}13 = 0{\cdot}05,$$

and this multiplying $C = 0{\cdot}044$ is too small for notice.

$$\beta_1 + \beta_2 - \beta_3 - \beta_4 = 4{\cdot}64 + 6{\cdot}85 + 0{\cdot}04 - 0{\cdot}04 = 11{\cdot}49,$$

$$\therefore D = -\frac{7^s{\cdot}51 + 0^s{\cdot}828}{11{\cdot}49} = -\frac{8^s{\cdot}338}{11{\cdot}49} = -0^s{\cdot}7222.$$

Apply this correction to the observations, we have for ϵ Orionis

Chronometer time of passing mean wire, -	$7^h\ 36^m\ 45^s{\cdot}54$
Correction for level error $0^s{\cdot}07 \times 0{\cdot}852$, - -	$-0\ {\cdot}06$
„ deviation $-0^s{\cdot}722 \times 0{\cdot}525$, -	$-0\ {\cdot}38$
„ collimation $-0^s{\cdot}04 \times 1{\cdot}0$, - -	$-0\ {\cdot}04$
Chronometer time of meridian passage, - -	$7^h\ 36^m\ 45^s{\cdot}06$
Right ascension of ϵ Orionis, - - - -	$5\ \ 47\ \ 53\ {\cdot}49$
Chronometer fast of sidereal time, - - -	$2\ \ 7\ \ 21\ {\cdot}56$

We next take α Orionis, the observations on which before recorded give

Chronometer time of α Orionis passing mean wire, - - - - - - - -	$7^h\ 55^m\ 15^s{\cdot}54$
Correction for level $= 0^s{\cdot}1 \times 0{\cdot}93$, - - -	$-0\ {\cdot}09$
„ deviation $= -0^s{\cdot}722 \times 0{\cdot}39$, -	$-0\ {\cdot}29$
„ collimation, illum. west $+0^s{\cdot}04 \times 1{\cdot}01$,	$+0\ {\cdot}04$
Chronometer time of α Orionis, meridian passage,	$7^h\ 55^m\ 15^s{\cdot}2$
Right ascension, - - - - - -	$5\ \ 47\ \ 53\ {\cdot}49$
Chronometer fast of sidereal time, - -	$2^h\ \ 7^m\ \ 21^s{\cdot}71$
„ „ by ϵ Orionis, - - -	$2\ \ 7\ \ 21\ {\cdot}56$
„ „ by means of equatorial stars,	$2^h\ \ 7^m\ \ 21^s{\cdot}635$

Chronometer time of α Geminorum passing mean wire, - - - - - -	$9^h\ 33^m\ 22^s{\cdot}27$
Correction for level $= 0^s{\cdot}12 \times 1{\cdot}181$, - -	$-0\ {\cdot}14$
„ deviation $= -0^s{\cdot}722 \times 0{\cdot}039$, -	$+0\ {\cdot}03$
„ collimation, illuminated end east, $-0^s{\cdot}044 \times 1{\cdot}182$, - - -	$-0\ {\cdot}05$

Chron. time of α Geminorum passing meridian,	9^h	33^m	$22^s{\cdot}11$
Right ascension of „ „ „	7	26	0·5
Chronometer fast of sidereal time, - -	2^h	7^m	$21^s{\cdot}61$
Chronometer time of 6 Cancri passing mean wire, - - - - - - -	10^h	2^m	$36^s{\cdot}6$
Correction for level $=0^s{\cdot}15 \times 1{\cdot}133$, - -			$-0{\cdot}17$
„ deviation $= -0^s{\cdot}722 \times 0{\cdot}043$, -			$-0{\cdot}03$
„ collimation, illuminated end west, $+0{\cdot}044 \times 1{\cdot}134$, - - -			$+0{\cdot}05$
Chronometer time of 6 Cancri passing the meridian, - - - - - -	10^h	2^m	$36^s{\cdot}45$
Right ascension of do. do., -	7	55	14·77
Chronometer fast of sidereal time, - -	2^h	7^m	$21^s{\cdot}68$
Do. do. mean of zenith stars,	2^h	7^m	$21^s{\cdot}645$
Hence at chronometer time 7^h 46^m it was fast of sidereal time, - - - -	2^h	7^m	$21^s{\cdot}635$
Do. do. 9^h 48^m	2	7	21·645
Do. do. 8^h 47^m	2^h	7^m	$21^s{\cdot}64$

Therefore on 1 December 1864, at sidereal time 6^h 40^m the chronometer was fast of sidereal time 2^h 7^m $21^s{\cdot}64$, gaining about $0^s{\cdot}005$ per hour, but this rate being given by an interval of two hours only is not to be depended on; but observations made on succeeding nights on the same stars will determine the rate accurately.

If we had set up a meridian mark such as that previously described, it would have been necessary to determine the point in which the plane of the meridian passing through the line of collimation of the transit telescope intersects the east and west shelf, so as to determine accurately where the light should be placed at night, and a mark sufficiently distinct to be seen through the transit telescope in the day time. Let d be the distance of the meridian mark from the transit expressed in feet, δ the angle which the straight line joining the point in which the middle wire intersects the east and west lamp shelf, and the centre of the transit station makes with the plane of the meridian passing through the transit; then $\delta \times d \times 12 \sin 1''$ = distance in inches measured along the lamp shelf from the point in which the middle wire intersects it, to the point in which the plane of the meridian cuts it. In the present case

we have found that when the illuminated end of the axis is east, the middle wire of the transit is $0^s{\cdot}03$ to the west of the line of collimation, which makes an angle of $0^s{\cdot}722$ with the plane of the meridian to the westward when the transit is pointing towards objects to the *north* of the zenith; and to the eastward when it is pointing at objects south of the zenith; consequently, when looking north the middle wire of the telescope points to an object in the horizon making an angle of $0^s{\cdot}752$ with the meridian to the westward, and when pointed to the south an angle of $0^s{\cdot}692$ to the eastward.

Suppose the meridian mark was placed 3,600 feet to the north of the transit instrument; now $\delta = 0^s{\cdot}752 = 11''{\cdot}28$. Therefore the distance along the shelf from the point in which the middle wire intersects it to that in which the meridian intersects it $= 11{\cdot}28 \times 3600 \times 12 \sin 1'' = 2{\cdot}363$ inches towards the west, and had the meridian mark been at the same distance to the southward of the transit the correction would have been 2·172 inches to the eastward along the lamp shelf from the point in which the middle wire of the transit telescope intersects it. When this correction has been made the deviation of the transit telescope can thereafter be tested, besides a very valuable line of true bearing will have been accurately established, and the further the meridian mark is from the transit station the more valuable this line of true bearing will be.

If two transit stations can be connected by a telegraph wire the most accurate and economical method is to use it. For this purpose a sidereal clock with a pendulum vibrating seconds, fitted to break the circuit every second, so as to note each second accurately on a cylinder revolving uniformly, on which also the observer can mark the instant at which a star crosses the wires of the transit telescopes by merely pressing down the key of a small instrument through which the electric current passes, and thereby breaking or making the circuit at any instant he pleases. This is such an accurate and expeditious mode of recording observations, that three more vertical wires can be placed equidistant between each of the seven vertical wires of the transit telescope, thus giving twenty-five instead of seven vertical wires, the passage of a star across each of which can in this way be observed with great precision; in every other respect the observations must be taken as before recommended, and after a complete and satisfactory series of observations have been made, the instruments and the observers must be changed and another complete set on the same stars, and as nearly as possible under the same circumstances, must be made.

CHAPTER V.

TRUE BEARINGS.

SURVEYING vessels generally depend on the theodolite and sextant, when the direction of the meridian or true bearings are to be determined; in this chapter it is proposed to point out the mode of using these instruments for the above purpose to the best advantage; but before describing the practice to be recommended, it will be advisable to discuss to some extent the theoretical considerations upon which it is founded.

First, we must determine what effect a given error in one of the elements defining the position of a heavenly body will produce on the true bearing.

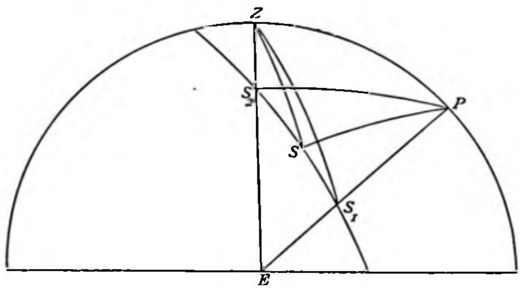


FIG. 14.

Secondly, we must find the position of the body when this error produces its smallest possible effect.

Thirdly, when the body is in a given position, we shall point out which of the elements had best be used for calculating the true bearing.

Lastly, when the observed altitude and hour angle of the body, as well as the latitude of the place, have been determined with different degrees of accuracy we shall show how to calculate

the probable sizes of the resulting errors in the respective true bearings calculated from them, taken two together, with the

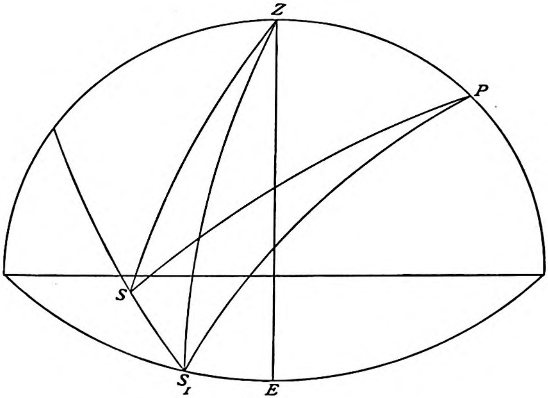


FIG. 15.

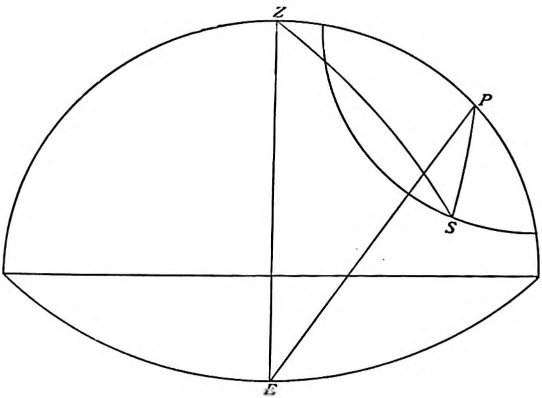


FIG. 16.

polar distance of the body; in order that the resulting true bearings may be meaned together in proportion to their respective values.

Let Z, Figs. 14, 15, and 16, be the zenith of the place, P the pole of the heavens, and S a heavenly body.

The triangle PZS determines the position of S, with respect to Z and P, and the sides and angles of this triangle are the elements of S's position, any three of which being known, the others can be calculated from them.

Let $ZP=c$, $ZS=z$, $PS=\Delta$, $ZPS=h$, $SZP=A$, $ZSP=S$, from which we obtain the following equations:

$$\sin A \,.\, \sin z = \sin h \,.\, \sin \Delta, \quad\ldots\ldots(1)$$

$$\sin A \,.\, \sin c = \sin S \,.\, \sin \Delta, \quad\ldots\ldots(2)$$

$$\cos z = \cos c \,.\, \cos \Delta + \sin c \,.\, \sin \Delta \,.\, \cos h, \quad\ldots\ldots(3)$$

$$\cos \Delta = \cos c \,.\, \cos z + \sin c \,.\, \sin z \,.\, \cos A, \quad\ldots\ldots(4)$$

$$\cos c = \cos z \,.\, \cos \Delta + \sin z \,.\, \sin \Delta \,.\, \cos S, \quad\ldots\ldots(5)$$

Let dh be a small change in the hour angle h, and dz, dA, and dS the consequent small changes in z, A, and S respectively.

From equation (3) we have

$$\sin z \,.\, dz = \sin c \,.\, \sin \Delta \,.\, \sin h \,.\, dh,$$

$$\therefore dz = \sin c \,.\, \sin A \,.\, dh, \quad\ldots\ldots(6)$$

$$= \sin \Delta \,.\, \sin S \,.\, dh, \quad\ldots\ldots(7)$$

From (4)
$$\cos A = \frac{\cos \Delta - \cos c \,.\, \cos z}{\sin c \,.\, \sin z},$$

$$\therefore \sin A \,.\, dA = \frac{\cos z \,.\, \cos \Delta - \cos c}{\sin c \,.\, \sin^2 z} dz, \quad\ldots\ldots(8)$$

$$\therefore \frac{dA}{dz} = -\frac{\cos S}{\sin c \,.\, \sin h} \quad\ldots\ldots(9)$$

Combining (7) with (8),

$$\frac{dA}{dh} = \frac{\cos z \,.\, \cos \Delta - \cos c}{\sin^2 z}, \quad\ldots\ldots(9b)$$

$$= -\frac{\cos S \,.\, \sin A}{\sin h} \quad\ldots\ldots(10)$$

Similarly
$$\frac{dS}{dz} = -\frac{\cos A}{\sin \Delta \,.\, \sin h}, \quad\ldots\ldots(11)$$

$$\frac{dS}{dh} = -\frac{\cos A \,.\, \sin S}{\sin h}. \quad\ldots\ldots(12)$$

Let dc be a small error in the colatitude of the place; then equation (4) gives

$$-\sin A \frac{dA}{dc} = \frac{\cos z - \cos c \,.\, \cos \Delta}{\sin^2 c \,.\, \sin z},$$

$$\therefore \frac{dA}{dc} = \frac{-\cos h}{\sin c \,.\, \sin h} = -\operatorname{cosec} c \,.\, \cot h \quad\ldots(13)$$

If from a small change in S's declination, Δ becomes $\Delta + d\Delta$, then in like manner

$$\frac{dA}{d\Delta} = \frac{\sin \Delta}{\sin c \,.\, \sin z},$$
$$= \frac{1}{\sin c \,.\, \sin h} \text{..........................(14)}$$

The error produced in the true bearing by an error in the observed altitude when using equation (4) for its determination is given by equation (9); from which we see that a small error a in the zenith distance produces an error $-\dfrac{a \cos S}{\sin c \,.\, \sin h}$ in the true bearing; and will therefore have its smallest possible effect when $\dfrac{\cos S}{\sin h}$ has its least numerical value.

When Δ is less than c (see Fig. 12), S can increase to 90°, and in this position of S, $\cos S = 0$, whilst $\sin h$ remains finite; and consequently a small error in the observed altitude, when S under this circumstance $= 90°$, produces no effect on the true bearing; when Δ is greater than c, as in Figs. 14 and 15, S will always be less than a right angle, and we must find the position of S when $\dfrac{\cos S}{\sin h}$ has its smallest value: to do this we make

$$\frac{\sin S}{\sin h}\,\frac{dS}{dh} + \frac{\cos S \,.\, \cos h}{\sin^2 h} = 0,$$

$$\therefore \text{ when } \cot h = -\tan S \frac{dS}{dh} \text{......................(15)}$$

$$\text{or } \cos h = \cos A \,.\, \sin S \,.\, \tan S \text{ ....(16)}$$

$\dfrac{\cos S}{\sin h}$ will have its smallest value numerically, and in this position of S a small error in the observed altitude will produce the smallest numerical error in the true bearing calculated from it, the colatitude and the polar distance.

We can eliminate A and S from equation (16), and obtain an equation involving h, Δ, and l; but practically it will be difficult to arrive at the value of h from the resulting expression, and it will be found more convenient to approximate to the value of h that will satisfy the condition given by equation (16). Here Δ is always greater than c; first, let $\Delta < 90°$ (as in Fig. 4), and S_1 the position of S when the angle $ZPS = 90°$; and S_2 when its true bearing $PZS_2 = 90°$; at S_1 the denominator of the fraction $\dfrac{\cos S}{\sin h}$ has its largest value, and at S_2 its numerator will have its smallest value: as S moves from S_1 to S_2 the angle

S continually increases, whilst the angle h continually diminishes; consequently, just after leaving S_1, cos S will be less than sin h, but will be decreasing more rapidly, and therefore $\frac{\cos S}{\sin h}$ will be diminishing as S moves towards S_2; on nearing S_2, however, where S has its largest size, S will increase very slowly, and therefore its cosine will diminish very slowly, whilst sine h is diminishing rapidly, and consequently $\frac{\cos S}{\sin h}$ will be increasing when S is near S_2; but immediately after leaving S_1 it was decreasing, consequently at some point S, between S_1 and S_2, its value will neither increase nor diminish, and will therefore have decreased to its smallest value.

$$\left.\begin{aligned} &\text{To determine } S_1 \text{ we have } \cos z_1 = \cos c \cdot \cos \Delta, \\ &\text{and } \sin S_1 = \frac{\sin c}{\sin z_1} \end{aligned}\right\} \qquad (17)$$

$$\text{and } S_2 \text{ is given by } \sin S_2 = \frac{\sin c}{\sin \Delta} \qquad (18)$$

Take S' the first approximate value of $S = \frac{S_1 + S_2}{2}$, and let h_2 be the hour angle at S_2; then as S moves from S_1 to S_2, its hour angle will diminish from 90° to h_2, and $90° - h_2$ will be the change in S's hour angle, whilst S changes from S_1 to S_2; therefore as a first approximation we make $\frac{dS}{dh} = \frac{S_1 - S_2}{90 - h_2}$, and substituting these values in equation (14):

$$\cot h' = \frac{S_2 - S_1}{90 - h_2} \cdot \tan \frac{S_1 + S_2}{z} \qquad (19)$$

from which the value of h' can be found. This will generally be sufficiently near the truth for practical purposes; but to obtain a closer approximation, calculate S' and A', the values of S and A corresponding to h', from the equation:

$$\left.\begin{aligned} \tan \frac{A' + S'}{2} &= \cot \frac{h'}{2} \cos \frac{\Delta - c}{2} \cdot \sec \frac{\Delta + c}{2} \\ \tan \frac{A' - S'}{2} &= \cot \frac{h'}{2} \sin \frac{\Delta - c}{2} \cdot \operatorname{cosec} \frac{\Delta + c}{2} \end{aligned}\right\} \qquad (20)$$

Take $\log \cos h_1' = \log (\cos A' . \sin S' . \tan S')$, and calculate from it A_1' and S_1'; let $\log \cos h' - \log \cos h_1' = D_1$, and $\log \cos h' - \log (\cos A_1' . \sin S_1' . \tan S_1') = D_2$.

Then, if $h_1'+x$ is the correct value of h, or the point where

$$\log\cos h-\log(\cos A\,.\sin S\,.\tan S)=0,$$

we have $$-x=\frac{(h'-h_1')D_2}{D_1-D_2}\text{ very approximately.........(21)}$$

Take the following example:—In latitude 45° 10′ N., find the hour angle of a star whose declination is 25° 20′ N., when the error in its true bearing calculated from c, Δ, z, arising from a small error in z, has its smallest numerical value. From equation (17)

$$\cos z_1=\cos(44°\ 50')\cos(64°\ 40');\ \sin S_1=\frac{\sin(44°\ 50')}{\sin z_1},$$

log cos (44° 50′), -	9·8507	log sin (44° 50′), -	9·8482
„ (64° 40′), -	9·6313	„ z_1, - - -	9·9790
log cos z_1(= 72° 20′ 15″)	9·4820	sin S_1(= 47° 44′), -	9·8692

From equation (18),

$$\sin S_1=\frac{\sin(44°\ 50')}{\sin(64°\ 40')}$$

log sin (44° 50′), -	9·8482
„ (64° 40′), -	9·9561
log sin S_2(= 51° 16′), -	9·8921

h_2 is given by $\cos h_2=\tan(44°\ 50')\cot(60°\ 40')$.

log tan (44° 50′) - -	9·9975
„ cot (64 40) - -	9·6752
„ cos h_2 (= 61° 55′ 15″) -	9·6727

Hence $$90°-h_2=28°\ \ 4'\ 45''=28°{\cdot}079,$$
$$S_1-S_2=\ 3\ \ 32\ \ \ =3\ {\cdot}567,$$
$$\frac{S_1+S_2}{2}=49°\ 30';$$

substituting these values in equation (19),

$$\cot h'=\frac{3{\cdot}567}{28{\cdot}079}\tan(49°\ 30'),$$

log 3·567 - - -	0·5523
AC log 28·079 - -	8·5515
log tan (49° 30′) - -	0·0685
log cot h'(=81° 32′ 30″) -	9·1723

Substituting these values in equation (20),

$$\tan\frac{A'+S'}{2}=\cot(40^\circ\ 46'\ 15'')\,.\,\cos(90^\circ\ 55')\,.\,\sec(54^\circ\ 45),$$

$$\tan\frac{A'-S'}{2}=\cot(40^\circ\ 46'\ 15'')\,\sin(9^\circ\ 55')\,\mathrm{cosec}\,(54^\circ\ 45'),$$

log cos (9° 55′), -	9·9935	log sin (9° 55′), - -	9·2361
log sec (54° 45′), -	0·2387	log cosec (54° 45′), -	0·0880
log C_1, - - -	0·2322	log C_2, - - -	$\bar{1}$·3241
log cot (40° 46′ 15″), -	0·0643	- - -	0·0643
log tan $\frac{A'+S'}{2}$, - -	10·2965	log tan $\frac{A'-S'}{2}$, - -	9·3884
∴ $\frac{A'+S'}{2}$, - -	= 63° 11′ 45″	∴ log cos A', - -	9·3541
		log sin S', - - -	9·8807
$\frac{A'-S'}{2}$, - -	= 14 44 30	log tan S', - -	0·0678
∴ A'	= 76° 56′ 15″	log cos h_1', - -	9·3826
S'	= 49 27 15		
		∴ h'_1,	= 78° 24′ 15″
log cot $\frac{h'}{2}$, - - -	0·0885	log cot $\frac{h'}{2}$, - - -	0·0885
log C_1, - - -	0·2322	log C_2, - - -	$\bar{1}$·3241
log tan $\frac{A_1'+S_1'}{2}$, -	10·3207	log tan $\frac{A_1'-S_1'}{2}$, -	9·4126

$$\therefore\ \frac{A_1'+S_1'}{2}=64^\circ\ 27'\ 45''\ \text{and}\ \frac{A_1'-S_1'}{2}=14^\circ\ 30'\ 0''.$$

∴ A_1'	= 78° 57′ 45″	log cos A_1', - -	9·2819
S_1'	= 49 57 45	log sin S_1' - - -	9·8840
		log tan S_1, - -	0·0757
log cos h', - - -	9·1723		9·2416
log cos h_1', - -	= 9·3026	log cos h_1', - -	9·3026
D_1	= −·1348	D_2	= 0·0610
h'	= 81° 32′ 30″		
h'_1	= 78 24 15		
$h'-h_1'$	= 3 8 15		

Substituting in (21),

$$x=(3^\circ\ 8'\ 15'')\times\frac{610}{1958}=58'\cdot 6.$$

$$h=h_1'+x=78^\circ\ 24'\cdot 25+58'\cdot 6=79^\circ\ 23' \text{ very approximately,}$$

and does not differ more than a few seconds of time from the exact value of h, which may easily be seen by carrying the approximation another step. We see that the first approximate value of h, viz. $81^\circ\ 32'\ 30''$ is within 9 minutes of time of the exact time, when $\dfrac{\cos S}{\sin h}$ has its smallest numerical value, and is quite near enough for all practical purposes, especially as in this position of S_1 $\dfrac{dA}{dz}$ changes very slowly.

Referring to equation (16), and taking the position of S such that $A=h$, and $\therefore Z=\Delta$,

and therefore $$\sin\frac{S}{2}=\sin\frac{c}{2}\,.\,\text{cosec}\,\Delta \qquad \text{.....................(22)}$$

let h' be the value of h when S is in this position, and A' and S' the corresponding values of A and S; putting these values in (22),

$$\sin\frac{S'}{2}=\sin\frac{c}{2}\sin\Delta,$$

$$\sin A'=\sin S'\frac{\sin\Delta}{\sin c},$$

$$h'=A'.$$

First, we observe when considering the condition $\cos h=\cos A\,.\,\sin S\,.\cos S$, that if $\sin S'\,.\tan S'>1$, h' will be too large; and if $\sin S'\,.\,\tan S'<1$, h' will be too small; and if the difference is not large, take $\cos h_1'=\cos A'\,.\sin S'\,.\,\tan S'$, and $h=\dfrac{h'+h_1'}{2}$ will give an approximate value of h quite near enough for general purposes.

Otherwise proceed to approximate to h in the same manner as before.

Taking the same example, we have

$$\sin\frac{S'}{2}=\sin(22^\circ\ 25')\ \text{cosec}\ (64^\circ\ 40'),$$

$$\sin A'=\sin S'\,.\,\sin(64^\circ\ 40')\ \text{cosec}\ (41^\circ\ 50').$$

log sin (22° 25′),	9·5813	log sin S',	9·8837
log cosec (64° 40′),	0·0439	log sin (64° 40′),	9·9561
log sin $\frac{S'}{2}$(=24° 57′ 15″),	9·6252	log cosec (44° 50′),	0·1516
∴ S'	=49° 54′ 30″	log sin A'(=78° 48′),	9·9916

log sin S',	9·8837		
log tan S',	0·0748	∴ h'	=70° 48′
log cos A',	9·2883	h'_1	=79 58
log cos h_1'(=79° 50′),	9·2468	∴ h	=79° 19′

may be taken as an approximate value of h sufficiently near for practice.

To obtain a closer approximation, calculate A_1' and S_1' from $h_1'=79°\ 50'$:

log cot $\frac{h_1}{2}$,	0·0775	log cot $\frac{h_1}{2}$,	0·0775
log C_1',	0·2322	log C_1,	$\bar{1}$·3241
log tan $\frac{A_1'+S_1'}{2}$,	0·3097	log tan $\frac{A_1'-S_1'}{2}$,	9·4016

$$\therefore \frac{A_1'+S_1'}{2}=63°\ 53'\ 30'' \text{ and } \frac{A_1'-S_1'}{2}=14°\ 9'\ 0''$$

$$\therefore A_1'=78°\ 2'\ 30'' \text{ and } S_1'=49°\ 44'\ 30:$$

log cos A_1',	=9·3164	log cos h',	9·2883
log sin S_1',	=9·8826	„ h_1',	9·2468
log tan S_1',	=0·7722		
		D_1	=0·0415
	9·2712		
log cos h_1',	9·2468	h'	=78° 48′
		h_1'	=79 50
D_2	=−·0244		
			62′

$$\therefore x=-\frac{244}{659}\,62'=-23';$$

$$\therefore h=h_1'+x=79°\ 50'-23'=79°\ 27' \text{ very approximately.}$$

This agrees very well with the result before obtained.

When using equation (1) to determine A, we have, considering h and z independent of each other,

$$\left.\begin{aligned} dA &= -\tan A\,.\cot z\,.dz \\ &= \tan A\,.\cot h\,.dh \end{aligned}\right\} \quad (23)$$

The first of these has its smallest value, disregarding sign,

when $z=90°$, so long as A is not 90° at the same time; also, when $z=A$ or $\pi-A$, $dA=dz$; when $z>A$ or $\pi-A$, $dA<dz$; but when $z<A$ or $\pi-A$, $dA>dz$, consequently, when using the zenith distance and hour angle to determine the true bearing, z should be $>A$, or $\pi-A$, and as large as possible.

The second has its smallest value when $h-90°$, so long as A is not 90° at the same time; but as h cannot be greater than A or $\pi-A$, except in the particular case $\Delta<c$ and $z>\Delta$, we see that generally, as far as this element is concerned, on the above conditions the nearer the hour angle is to 6^h the better.

We also observe that when the body is near the prime vertical this equation ought not to be used, or, in other words, the hour angle and zenith distance ought not to be used to determine the true bearing; A, or $180°-A$, ought not generally to exceed 45° when using equation (1).

The three parts of the triangle PZS (Figs. 14, 15, and 16) which depend upon observations made at the place, or at some other place and reduced thereto, are ZP, ZS, and ZPS, of which ZP is derived from the observed altitude of S made on the spot, ZP upon observations made to determine the latitude either made at the place, or, if on board a vessel in motion, at some other place, and combined with her dead reckoning; the angle ZPS upon the time carried by a good chronometer, whose error on apparent time has been determined from observations made at the place, or, if on board a vessel in motion, at some other place, and combined with her dead reckoning. These are all liable to errors the sizes of which can be estimated with different degrees of accuracy according to circumstances, and the accuracy of their relative proportions as to size will depend in a great measure upon the experience, tact, and judgment of the director of the survey.

Upon the before-mentioned considerations let α be the probable size of the error in the zenith distance of the body, β that of the hour angle of the body, and γ that of the latitude of the place. Let also $dA(z)$ be the error in the true bearing, caused by a small error α in the zenith distance; $dA(h)$ that arising from a small error β in the hour angle; and $dA(c)$ that from a small error γ in the colatitude of the place; then when h and c, or z and c, are used together with Δ for calculating the true bearing, we have

$$dA(z) = -\frac{\alpha \cos S}{\sin c \,.\, \sin h}, \qquad (24)$$

$$dA(h) = -\frac{\beta \cos S \,.\, \sin A}{\sin h}, \qquad (25)$$

$$dA(c) = -\gamma \operatorname{cosec} c \,.\, \cot h, \qquad (26)$$

but when h, z and Δ are used for determining A,

$$dA(z) = -\alpha \,.\, \tan A \,.\, \cot z, \ldots\ldots\ldots\ldots\ldots (27)$$

$$dA(h) = -\beta \,.\, \tan A \,.\, \cot h, \ldots\ldots\ldots\ldots\ldots (28)$$

When z and c are used with Δ to determine A, expressing the whole error depending on α and γ by $dA(z, c)$, we see, disregarding the sign and only introducing the condition that $dA(z)$ and $dA(c)$ may have the same or different signs, each being equally probable, that considering only size,

$$dA(z,c) = \frac{\alpha \cos S \pm \gamma \cos h}{\sin c \,.\, \sin h}.$$

The probable size of this error will be the half sum of the two quantities $\frac{\alpha \cos S + \gamma \cos h}{\sin c \,.\, \sin h}$ and $\frac{\alpha \,.\, \cos S - \gamma \cos h}{\sin c \,.\, \sin h}$, which will consequently be of the same size as the larger of the two.

Therefore the most probable size of the error $dA(z,c)$ will be that of the larger of the two quantities $\frac{\alpha \cos S}{\sin c \,.\, \sin h}, \frac{\gamma \,.\, \cos h}{\sin c \,.\, \sin h}$; consequently when $\alpha \cos S > \gamma \cos h$, we must take the former quantity as representing the most probable size of $dA(z,c)$; but when $\alpha \cos S < \gamma \cos h$, the latter size must be used to express that of $dA(z,c)$. In a similar manner the most probable size of $dA(h,c)$ is $\frac{\beta \cos S \,.\, \sin A}{\sin h}$, or $\frac{\gamma \,.\, \cos h}{\sin c \,.\, \sin h}$, according as $\beta \cos S \,.\, \sin A \,.\, \sin c$ is $>$ or $< \gamma \cos h$. And the most probable size of $dA(z, c)$ is $\alpha \tan A \,.\, \cot z$ or $\beta \tan A \cot h$, according as $\alpha \cot z >$ or $< \beta \cot h$.

Having determined the value of $dA(z,c)$, $dA(h,c)$, and $dA(z,h)$ in the foregoing manner,

let $$dA(z, c) = \frac{1}{p}, \; dA(c,h) = \frac{1}{q}, \; dA(z,h) = \frac{1}{r}.$$

Then if $A(z,c)$ represents the value of A as determined by means of the equation

$$\tan \frac{A(z,c)}{2} = \frac{\sin(\Sigma - z) \,.\, \sin(\Sigma - c)}{\sin \Sigma \,.\, \sin(\Sigma - \Delta)}, \ldots\ldots\ldots\ldots (29)$$

where $\Sigma = \frac{\Delta + c + z}{2}$, $A(c,h)$ the value of A calculated from Δ, c and h by means of equations (23), and $A(z,h)$ that calculated from A, z, and h by means of equation (1).

Meaning these in the inverse proportion of the probable

sizes of their respective errors, we find the most probable value of A derived from its three values thus found to be

$$=\frac{pA(z,c)+qA(c,h)+rA(z,h)}{p+q+r}.$$

Take the following example:—Ship in latitude 45° 8′ 20″N, sun's declination 20° 28′ 50″N; apparent time at ship determined by means of a good chronometer, whose error on apparent time by observations on the sun when near the prime vertical, brought on by the dead reckoning, being $7^h\ 53^m\ 40^s$ A.M., and zenith distance of the sun's centre from its observed altitude corrected being 55° 45′ 15″; the probable sizes of the errors α, β, and γ being in the proportion respectively of

$$\frac{\alpha}{\beta}=\frac{3}{2} \text{ and } \frac{\alpha}{\gamma}=\frac{3}{2}.$$

$$\begin{array}{rlll} \text{Here we have } \Delta= & 69^\circ & 31' & 10'' \\ c= & 44 & 51 & 40 \\ z= & 55 & 45 & 15 \\ \therefore \quad \Sigma= & 85 & 4 & 2{\cdot}5 \\ \Sigma-z= & 29 & 18 & 47{\cdot}5 \\ \Sigma-c= & 40 & 12 & 22{\cdot}5 \\ \Sigma-\Delta= & 15 & 32 & 52{\cdot}5 \end{array} \qquad \begin{array}{rlll} \Delta+c= & 114^\circ & 22' & 50'' \\ \therefore \frac{\Delta+c}{2}= & 57 & 11 & 25 \\ \Delta-c= & 24 & 39 & 30 \\ \frac{\Delta-c}{2}= & 12 & 19 & 45 \end{array}$$

From equation (29),

$$\begin{array}{lr} \log\sin\ (29^\circ\ 18'\ 47''{\cdot}5)\ \text{- -} & 9{\cdot}689827 \\ \quad ,,\quad (40\ \ 12\ \ 22)\ \text{- -} & 9{\cdot}809923 \\ \log\operatorname{cosec}\ (85\ \ 4\ \ 2{\cdot}5)\ \text{- -} & 0{\cdot}001612 \\ \quad ,,\quad (15\ \ 32\ \ 52{\cdot}5)\ \text{- -} & 0{\cdot}571793 \\ \hline \log\tan^2\frac{A(z,c)}{2},\ \text{- - - -} & 20{\cdot}073155 \\ \hline \log\tan\frac{A(z,c)}{2}(=47^\circ\ 24'\ 35'')\ \text{-} & 10{\cdot}036577 \end{array}$$

$$\therefore \qquad A(z,c)=94^\circ\ 49'\ 10''.$$

From equation (23),

$$\begin{array}{lrlr} \log\cot\ (2^h\ 3^m\ 10^s)\ \text{- -} & 0{\cdot}224810 & \text{do.} & 0{\cdot}224810 \\ \log\cos\ (12^\circ\ 19'\ 45)\ \text{- -} & 9{\cdot}989867 & \log\sin\text{ do.} & 9{\cdot}329454 \\ \log\sec\ (57\ \ 11\ \ 25)\ \text{- -} & 0{\cdot}266120 & \log\operatorname{cosec}\text{ do} & 0{\cdot}075476 \\ \hline \tan\frac{A(c,h)+S}{2}\ \text{- - -} & 0{\cdot}480797 & \tan\frac{A(c,h)-S}{2} & 9{\cdot}629740 \end{array}$$

$$\therefore \frac{A(c,h)+S}{2}=71^\circ\ 42'\ 36'';\ \frac{A(c,h)-S}{2}=23^\circ\ 5'\ 23''.$$

$$\therefore \quad A(c,h)=94^\circ\ 47'\ 59'' \text{ and } S=58^\circ\ 37'\ 13''.$$

From equation (1),

$$\sin A(z, h) = \sin (4^h\, 6^m\, 20^s) \sin (69^\circ\ 31'\ 10'') . \operatorname{cosec} (55^\circ\ 45'\ 15'')$$

log sin	$(4^h\ 6^m\ 20^s)$ - -	9·944241
„	$(69^\circ\ 31'\ 10'')$ - -	9·971643
log cosec	$(55\ \ 45\ \ 15\ )$ - -	0·082689
log sin $A(z, h)$	$(=94^\circ\ 38'\ 30'')$ -	9·998573

$$\gamma \cos h = \frac{3a}{2} \cos h = \frac{3a}{2} \cos (4^h\ 6^m\ 20^s),$$

$$a \cos S = a . \cos (58^\circ\ 37').$$

log 3, - - -	0·4771	log cos $(58^\circ\ 37')$, -	9·7166
AC log 2, - - -	$\bar{1}$·6990		
log cos $(4^h\ 6^m\ 20^s)$, -	9·6775		
	9·8536	$\therefore \gamma \cos h > a \cos S$.	
log cosec c, - -	0·1516		
log cosec h, - -	0·0554		
log (1·15), - - -	0·0606	$\therefore dA(z, c) = 1{\cdot}15 \times a$.	

Since $\beta > a$, $\beta \cos S . \sin A \sin c$ is $< \gamma \cos h$, and therefore also

$$dA(h, c) = 1{\cdot}15 \times a,$$

$$\beta \cot h = \frac{2a}{3} \cot (4^h\ 6^m\ 20^s),$$

$$a \cot z = a \cot (55^\circ\ 45'\ 15''):$$

the latter being the larger of the two we have

$$dA(h, z) = \text{numerically } a . \tan (94^\circ\ 48') \cot (53^\circ\ 45'\ 15''),$$

log tan $(94^\circ\ 48')$, -	11·0759
log cot $(55^\circ\ 45'\ 15'')$, -	9·8330
log 8·11, - -	0·9089 $\quad \therefore dA(z, h) = 8{\cdot}11 \times a$.

$$\frac{1}{1{\cdot}15} = 0{\cdot}87 \text{ and } \frac{1}{8{\cdot}11} = 0{\cdot}123.$$

$$A = \frac{A(z, h) \times 0{\cdot}123 + A(c, h) \times 0{\cdot}87 + A(c, z) \times 0{\cdot}87}{1{\cdot}863}.$$

$$A\ (z, h) = 94^\circ\ 38'\ 30'',$$

$$A\ (c, h) = 94\ \ 38\ \ 30 + \ 9'\ 29'',$$

$$A\ (z, c) = 94\ \ 38\ \ 30 + 10\ \ 40,$$

$$\therefore \quad A = 94\ \ 38\ \ 30 + \frac{(20'\ 9'') \times 290}{621},$$

$$= 94\ \ 38\ \ 30 + 9'\ 24'' = 94^\circ\ 47'\ 54.$$

which is the most probable value which can be derived from the observations.

Referring to equation (10) we find that

$$\frac{dA}{dh}=-\frac{\cos S.\sin A}{\sin h}.$$

If $\Delta>c$ there is one position of S where the angle $S=90°$, whilst A and h will give a finite value for $\frac{\sin A}{\sin h}$ in this position, therefore $\frac{dA}{da}=0$, and here a small error in h will have no effect on the value of A; but if $\Delta>c$, S will always be less than 90°; to determine the smallest value of $\frac{dA}{dh}$ in this case from equation (9*b*), we have $\frac{dA}{dh}=\frac{\cos z.\cos\Delta-\cos c}{\sin^2 z}$: when this is a minimum, we must have

$$\frac{\cos\Delta}{\sin z}-\frac{2\cos z\,(\cos z.\cos\Delta-\cos c)}{\sin^3 z}=0,$$

$$\therefore\qquad \cos\Delta\,(1+\cos^2 z)-2\cos z.\cos c=0,$$

$$\cos^2 z=2\frac{\cos c}{\cos\Delta}\cos z+1=0,$$

$$\therefore\qquad \cos z=\frac{\cos c}{\cos\Delta}-\sqrt{\frac{\cos^2 z}{\cos^2 c}-1}\;\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

The positive sign of the root not being admissible, so long as $\Delta<90°$, we can always find a value of $z<90°$ which will satisfy (30), and give the position of S when $\frac{dA}{dh}$ has its smallest value; but when $\Delta>90°$, the value of the right-hand side of (30) will always be negative, and the value of z will therefore be $>90°$, or the body below the horizon, and in this case, therefore, the nearer the body is to the horizon the smaller $\frac{dA}{dh}$ will be.

If v be the altitude of the body when on the prime vertical $\sin v=\frac{\cos\Delta}{\cos c}$, therefore, substituting in (30),

$$\cos z=\operatorname{cosec} v-\cot v$$

$$=\tan\frac{v}{2}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

Therefore, when $\Delta<90°$ but $>c$, $\frac{dA}{dh}$ has its smallest value

when the cosine of the zenith distance of the body is equal to the tangent of half its altitude when on the prime vertical.

From equations (6) and (7)

$$\left.\begin{aligned}\frac{dh}{dz} &= \operatorname{cosec} c \,.\, \operatorname{cosec} A \\ &= \operatorname{cosec} \Delta \,.\, \operatorname{cosec} S\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(32)$$

also, if dh be the small change in the hour angle consequent on a small change dc of the colatitude, from equation (3)

$$0 = (\cos c \,.\, \sin \Delta \,.\, \cos h - \sin c \,.\, \cos \Delta)dc,$$
$$- \sin c \,.\, \sin \Delta \,.\, \sin h \,.\, dh,$$
$$\therefore \frac{dh}{dc} = \frac{\cos c \,.\, \sin \Delta \,.\, \cos h - \sin c \,.\, \cos \Delta}{\sin c \,.\, \sin \Delta \,.\, \sin h},$$
$$= \frac{\cot c \,.\, \cos h - \cot \Delta}{\sin h},$$
$$= -\frac{\cos A}{\sin S} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33)$$

We will now take an example to show the practical application of the foregoing:

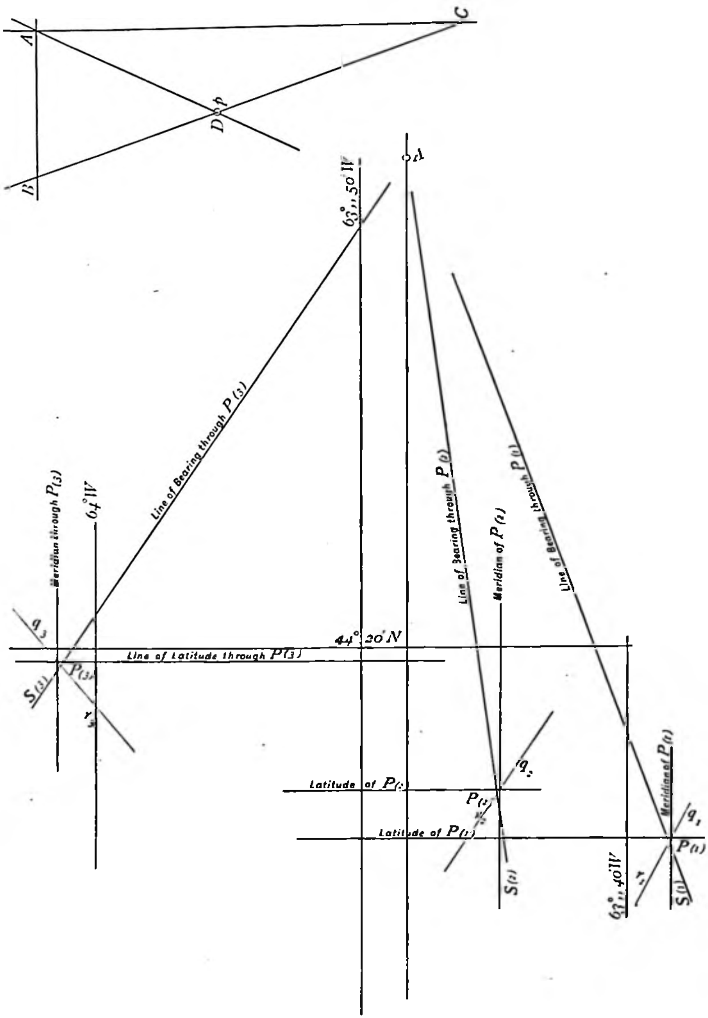
4 August 1858, ship in latitude 44° 15′ N. and longitude 63° 38′ W. approximately determined, was steered carefully to make a true course N. 73° W.; having previously determined that the sun's altitude at 6 A.M. would be about 12°, and when on the prime vertical about 25½°; the patent logs were placed in the water, and the following observations were taken at No. 1 position of the ship, and denoted by P(1) (see Fig. 13).

Chronometer.	Alt. of Sun's LL.	Bearing.
$11^h\ 21^m\ 24^s$.	11° 54′ 20″	Station A. 97° 15′ sun's NL.
	Index error −40″	No index error

the above being the mean of five observations.

The error of the chronometer at the time of observation was $1^h\ 0^m\ 55^s$ fast of Greenwich mean time, and gaining daily $1^s{\cdot}2$; the index error of the sextant with which the sun's altitude was observed was −40″, and the same during the succeeding observations. The sextant with which the sun's bearing was observed was the same throughout the day, and had no index error. At position 2, denoted by P(2), the following observations were taken, mean of patent log readings 4′·78:

Chronometer.	Alt. of Sun's LL.	Bearing.
$0^h\ 36^m\ 15^s$	25° 3′ 10″	A. 96° 46′ sun's NL.



A
B
C
D
p
63° 50' W
A
Line of Bearing through P(3)
Meridian through P(3)
64° W
44° 20' N
Line of Latitude through P(3)
q3
P(3)
S(3)
r3
Line of Bearing through P(2)
Meridian of P(2)
Line of Bearing through P(1)
Meridian of P(1)
q2
Latitude of P(2)
P(2)
Latitude of P(1)
S(2)
63° 40' W
q1
P(1)
S(1)

To face p. 140.

At position 3, denoted by P(3), the reading of patent logs 17′·28, observations taken, the mean of 5 given below:

Chronometer.	Alt. of Sun's LL.	Bearing.
3^h 49^m 11^s	56° 24′ 20	A. 97° 22′ sun's NL.

The sun's meridian altitude observed at noon with the same sextant having the same index error of −40″, gave altitude of sun's LL. 62° 41′ 57″, the mean of the patent logs gave distance run from P(1) 23·28 miles, the height of the eye above the sea was 18 feet.

It was estimated that the probable size of the error in the observed altitude of the sun was ±0′·3; that of the difference of latitude by D.R. to be ±0′·1 per hour, and that of the difference of longitude ±0′·1 per hour.

At Greenwich mean noon, 4 August, the sun's declination was 17° 16′ 42″ N., hourly change 42″·3 S.; also, the sun's semi-diameter at the same time was 15′ 48″; the equation of time was 5^m 49^s·08 to be subtracted from mean time; hourly difference 0^s·23 decreasing.

Course.	Dist.	Diff. Lat.	Prob. Size of Error.	Dep.	Diff. Long.	P. Size of E.
From (1) to (2), N. 73° W.,	4′·78	1′·4	+ or − 0′·15	4′·57	6′·4	±0′·13
(2) to (3), - -	12·5	3·65	0·31	11·95	16·6	±0·31
(3) to noon, - -	6·0	1·75	0·14			

Latitude at noon by sun's meridian altitude, -	44° 21′·33 N.
Difference of latitude, P(3), - - - -	−1·75 S.
Latitude P(3), probable size of error ±0′·31, -	44° 19′·58 N.
Difference of latitude, P(2), - - - -	−5′· 4 S.
Latitude P(2), probable size of error ±0′·46, -	44° 15′·93 N.
Difference of latitude, P(1), - - - -	−6′· 8 S.
Latitude P(1), probable size of error ±0′·6, -	44° 14′·53 N.

We shall take each quantity to the nearest tenth of a mile, and use four places of figures in the logarithms as quite near enough for ordinary purposes. In order to determine the hour angle corresponding to each position, we commence with the observations made at P(2), when the sun was very close to the prime vertical.

Chronometer time,	$0^h\ 36^m\ 15^s$	Sun's dec., G.M.N., 4 August,	17° 16′ 42″ N.
Fast Green. M.T.,	1 0 55	Change in 25″,	+18
Green. M.T., 3 Aug.,	$23^h\ 35^m\ 20^s$	Declination,-	17° 17′ 0″ N.
Ship's 2d Pos.,	19 20 21		
		Observed alt., L.L.	25° 3′ 10″
Longitude P(2),	$4^h\ 14^m\ 59^s$	Index error, -	− 40
		Dip, - -	− 4 11
		Ref. less Plx., -	− 1 54
Polar dist., - -	72° 43′	Semi-diam., -	+15 48
Zenith „ - -	64 47·8		
P(2), colatitude, -	45 44·1	True alt. ⊙ -	25° 12′ 13″
Sum, - -	183° 14′·9	Z. D., - -	64° 47′ 47″
Σ, - - -	91° 37′·4	log cosec, - -	0·0002
$\Sigma - z$, - - -	26 49·6	„ - -	0·3455
$\Sigma - c$, - - -	45 53·3	log sin, - - -	9·8561
$\Sigma - \Delta$, - - -	18 54·4	„ - - -	9·5106
			19·7124
		log tan $\frac{h}{2}$, - -	9·8562

$$\therefore \frac{h}{2} = 35^\circ\ 41'$$
$$h = 71\ \ 22$$
$$= 4^h\ 45^m\ 38^s$$

Apparent time at ship, - - - -	$19^h\ 14^m\ 32^s$
Equation of time, - - - -	+5 49
Mean time at P(2), - - - -	$19^h\ 20^m\ 21^s$

To determine the probable size of h's error, we have—

$$\text{Equation (32) gives } \frac{dh}{dz} = \text{cosec } c \,.\, \text{cosec } A,$$

$$\text{and} \quad \text{(33) gives } \frac{dh}{dc} = -\frac{\cos A}{\sin S}.$$

Since $A = 90°$ very approximately, and $\sin S$ a positive quantity, $\frac{dh}{dc} = 0$, The probable size of $dz = \pm 0'{\cdot}3$, observing that cosec $A = 1$ very approximately :

$$dh = \pm 0{\cdot}3 \text{ cosec } (45^\circ\ 44'),$$

log 0·3	$=\bar{1}{\cdot}4771$
log cosec (45° 44′),	0·1448
0′·42, - - -	$\bar{1}{\cdot}6219$

Hence $h = 4^h\ 45^m\ 28^s$, probable size of error $= \pm 1^s{\cdot}7$.

To calculate the true bearing of A from position P(2):

log sin	26° 49′·6	- - - -	9·6545
„	45 53·3	- - - -	9.8561
log cosec	91 37·4	- - - -	0·0002
„	18 54·4	- - - -	0·4894
			0·0002
log tan $\frac{A}{2}$ (= 45° 0′·5),		- - - -	0·0001

$\therefore A(z, c) =$ N. 90° 1′ E.

Probable size of c's error $= \pm 0'{\cdot}46$.

„ z's $= \pm 0'{\cdot}3$.

Hence $dA(c) = \pm 0'{\cdot}46 \times \text{cosec } (45^\circ\ 44)\ \cot\ (71^\circ\ 22'),$

$dA(z) = \pm 0'{\cdot}3 \times \text{cosec } (71^\circ\ 22)\ .\ \cos S\ .\ \text{cosec } (45^\circ\ 44'),$

and consequently $dA(z, c)$ is equal to the (numerically) larger of the two; before we can calculate the latter we must know the value of S, which will be given in calculating $A(h, c)$, as follows:

$$\Delta = 72^\circ\ 43', \text{ and } c = 45^\circ 44'{\cdot}1, \text{ and } \frac{h}{2} = 35^\circ\ 41';$$

$$\therefore \frac{\Delta + c}{2} = 59^\circ\ 13'{\cdot}5, \frac{\Delta - c}{2} = 13^\circ\ 29'{\cdot}5,$$

log cot $\frac{h}{2}$ - -	0·1438	do.	0·1438
log cos (13° 29′·5), -	9·9878	log sin do.	9.3680
log sec (59° 13′·5), -	0·2910	log cosec do.	0·0659
tan (69° 18′), - -	0·4226	tan (20° 43″)	9·5777

$$\frac{A + S}{2} = 69^\circ\ 18',$$

$$\frac{A - S}{2} = 20^\circ\ 43';$$

$$\therefore \quad A(h, c) = 90^\circ\ 1' \qquad S = 48^\circ\ 35',$$

and $dA(h) = \pm 0'{\cdot}42\ .\ \cos\ (48^\circ\ 35')\ .\ \text{cosec } (71^\circ\ 22);$

log 0·46, - -	$\bar{1}$·6628	log 0·3, - -	$\bar{1}$·4771	log 0·42, -	$\bar{1}$·6232
log cot (71° 22′),	9·5279	log cosec, -	0·0234	log cosec, -	0·0234
log cosec (45° 44′),	0·1448	- - -	0·1448	log cos (48° 35′),	9·8205
		log cos (48° 35′),	9·8205		
log (0·22), - -	$\bar{1}$·3355	log (0·29) -	= $\bar{1}$·4658	log (0·29),	= $\bar{1}$·4671

$$
\begin{aligned}
\therefore\quad \text{probable size of } dA(c) &= \pm 0'{\cdot}22 \\
\text{,,}\quad \text{,,}\quad dA(z) &= \pm 0\ {\cdot}29 \\
\text{,,}\quad \text{,,}\quad dA(h) &= \pm 0\ {\cdot}29 \\
\therefore\quad \text{,,}\quad \text{,,}\quad dA(h,c) &= \pm 0'{\cdot}29 \\
\text{,,}\quad \text{,,}\quad dA(z,c) &= \pm 0\ {\cdot}29
\end{aligned}
$$

From which it follows that $A(h, c)$ and $A(z, c)$ are equally good; we also observe, the A being very nearly 90°, the error resulting from calculating $A(h, c)$ will be so large as to render it of no value when compared with $A(h, c)$ and $A(z, c)$. $A(h, c)$ and $A(z, c)$ give exactly the same result, and it is not necessary to calculate their relative values, as their mean will be the same whatever those values might have been; we have merely calculated them to show how it should be done:

Hence, P(2), true bearing of sun's centre is N. 90° 1′ E.

Latitude of P(2),	- 44° 15′·93 N.,	probable error	± 0′·46
Longitude ,,	4h 14m 59s	,,	± 1s·7
	63 44·73 W.,	,,	± 0 ·42

Latitude of A. as given by the shore survey, - 44° 33′.78 N.
Longitude ,, ,, ,, - 64° 48′·32 W.

The inclination of the meridians through these two places is therefore about 2′·4, and their Mercatorial meridians will be inclined to each of them respectively about 1′·2.

From P(2), - A. 96° 45′ Sun's NL.
+15 ·8 Semi-diameter.

A. 97° 0′·8 Sun's (centre altitude 25° 12′·2)

log cos (97° 0′·8), - - -	9·0867
log sec (25° 12′·2), - - -	0·0434
log cos hor. ang. (97° 45′·2), -	9·1301

True bearing of the sun from P(2), - -	N. 90° 1′ E.
Horizontal angle ,, - -	97° 45′·2 Sun.
True bearing of A. from P(2), - - -	N. 7° 44′·2 W.
Inclination of Mercatorial to true meridian,	+1′·1

Mercatorial bearing, - - - - -	N.	7° 45′·3 W.
" P(2) from A., - -	S.	7° 45′·3 E.

We next determine the Mercatorial bearing of P(1) from A. as follows:

We have, by D.R., P(1) east of P(2) 6′·4, probable size of error $=0'\cdot13$.

Chronometer fast of apparent time, P(2), losing on apparent time 4^s per diem, -	5^h	21^m	43^s
∴ Faster at 11^h 21^m than at 12^h 36^m, - -			0·2
Slower from different longitudes, - -			25·6
Fast of A. G.T. P(1) at 11^h 21^m 24^s, - -	5^h	21^m	$17^s\cdot6$
Chronometer time, - - - - -	11	21	24
Apparent time at P(1) at chronometer time 11^h 21^m 24^s, - - - - - -	6	0	6·4

∴ $h=5^h\ 59^m\ 55^s\cdot6$, probable size of error $\pm\ 0^s\cdot42$.
$=89°\ 58'\cdot9$

From equation (1) $\sin A(h,z)=\sin(89°\ 58'\cdot9)\sin\Delta\,.\,\text{cosec}\,z$. The observations give

Chronometer time,	11^h 21^m 24^s	Obs. alt. sun's LL.,	11° 54′ 20″
Fast Greenw. M.T.,	1 0 55	Index error, -	−40
		Dip, - - -	−4 11
Green. M.T., 3 Aug.,	22^h 20^m 29^s	Refr. less parallax,	−4 20
		Semi-diameter, -	15 48
4 August, G.M.N., sun's dec.,	17° 16′ 42″N.	True alt., - -	12° 0′ 57″
Cor. 1^h 40^m,	+1 6	or - -	12° 0′·95
Corrected dec.,	17° 17′ 48″N.	Z.D. - - -	77° 59′·05
or - -	17° 17′·6 N.		
P. distance, -	72° 42′·4		

log sin (89° 58′·9), -	0·0000
" (72° 42·4), -	9·9799
log cosec (77° 59′), -	0·0096
log sin (77° 28′·3), -	9·9895

∴ $A(h,z)=$ N. 77° 28′·3 E.

Now $dA(h) = \pm 0'\cdot 42 \tan(77°\ 28') \cot(89°\ 58'\cdot 9)$, and is too small to be noticed.

$$dA(z) = \pm 0'\cdot 3 \tan(77°\ 28') \,.\, \cot(77°\ 59'),$$

log 0·3, - -	$\bar{1}$·4771
log cot (77° 59′), -	9·3281
log tan (77° 28′), -	0·6531
log 0·29, - -	$\bar{1}$·4583

Hence, $A(h, z) = $ N. 77° 28′ 3″ E., probable error, $\pm 0'\cdot 29$.

$\log\cot\frac{h}{2}(=49°\ 59'\cdot 5)$, -	0·0002	do., - - -	0·0002
$\log\cos\frac{\Delta - c}{2}(=13°\ 28'\cdot 5)$,	9·9879	log sin, - - -	9·3674
$\log\sec\frac{\Delta + c}{2}(=59°\ 14')$,	0·2911	log cosec, - - -	0·0659
tan (62° 16′), - -	0·2792	tan (15° 11′), - -	9·4335

$$\therefore \frac{A+S}{2} = 62°\ 16',\ \frac{A-S}{2} = 15°\ 11';$$

$$\therefore\ A(h, c) = \text{N. } 77°\ 27' \text{ E. and } S = 47°\ 5'.$$

To determine the probable size of its error we have

$$dA(h) = \pm 0'\cdot 42 \cos(47°\ 5') \sin(77°\ 27') \operatorname{cosec}(89°\ 59'),$$

log cos (47° 5′), - - -	9·8331
log sin (77° 27′), - - -	9·9895
log 0·42, - - - -	$\bar{1}$·6232
log 0·28, - - - -	$\bar{1}$·4458

$$dA(c) = \pm 0'\cdot 6 \operatorname{cosec}(45°\ 45') \cot(89°\ 59')$$

is too small to have any practical effect.

$\therefore\ A(h, c) = $ N. 77° 27′ E., probable size of error, $\pm 0'\cdot 28$.

Now	$z =$	77° 59′·05
	$\Delta =$	72 42·4
	$c =$	45 45·5
Sum		= 196° 27′

$\Sigma =$	98° 13′·5	log cosec, - -	0·0045
$\Sigma - z =$	20 14·5	log sin, - - -	9·5390
$\Sigma - c =$	52 28	log sin, - - -	9·8993
$\Sigma - \Delta =$	25 31·1	log cosec, - -	0·3658

$\therefore$ sum of logs, - - - 19·8086

log tan 38° 44′·25, - - - - 9·9043

$\therefore$ $A(z, c) =$ 77° 28′·5.

We have seen that $dA(c)$ is too small, and therefore it only remains to calculate

$$dA(z) = \pm 0'{\cdot}3 \times \cos(47° \; 5') \operatorname{cosec}(45° \; 45') \operatorname{cosec}(89° \; 59').$$

log 0·3, - - - -	$\bar{1}$·3010
log cos (47° 5′), - - -	9·8331
log cosec (45° 45′), - - -	0·1448
log 0·19, - - - -	$\bar{1}$·2789

Hence $A(z, c) =$ N. 77° 28′·5 E., probable size of error, ± 0′·19. Taking the value of each of these determinations to be in inverse proportion of the probable size of their errors we have

$A(h, z) =$ N. 77° 28′·3 E., value, 3·45
$A(h, c) =$ N. 77 27 E., „ 3·57
$A(z, c) =$ N. 77 28·5 E., „ 5·26

$$\therefore \quad A = \text{N. } 77° \; 27' \text{ E.} + \frac{1'{\cdot}3 \times 3{\cdot}45 + 1'{\cdot}5 \times 5{\cdot}26}{12{\cdot}28} \text{E.},$$
$$= \text{N. } 77 \;\; 27 \;\; \text{E.} + 1' \text{ E.} = \text{N. } 77° \; 28' \text{ E.}$$

The most probable value of the sun's true bearing from P(1) to be derived from the observations is

Latitude P(1), - -	44° 14′·53 N.
Longitude, - - -	63 38·35 W.

Therefore, inclination of the meridians A. and P(1) = 6′·9, and the Mercatorial meridians to either = 3′·45;

$\therefore$ True bearing by observation,	A. 97° 15′	Sun's LL.
	+ 15·6	Semi-diam.
	A. 97° 30′·6	Sun's centre.

Altitude sun's centre, 12° 1′.

log cos 97° 30′·6, - - -	9·1164
log sec 12 1, - - - -	0·0096
log cos horizontal angle(= 97° 40′·9),	9·1260

Therefore, at P(1),

Horizontal angle, - -	A. 97° 40′·9 Sun's centre.
True bearing sun's centre, -	N. 77 28 E.

Therefore true bearing A., -	N. 20° 12′·9 W.
Cor. for Mercatorial bearing,	+3·45
∴ Mercatorial bearing, -	N. 20° 16′·35 W.

Therefore, Mercatorial bearing of P(1) from A. is S. 20° 16′·35 E.

To determine the Mercatorial bearing of the ship's third position, P(3), from A. we have

Chronometer,	3^h 49^m 11^s	Obs. alt. sun's LL.,	56° 24′ 20″
Fast Green. M.T.	1 0 55	Index error, -	−40
Green., 4 Aug.,	2^h 48^m 16^s	Dip, - - -	−4 11
Sun's dec., 4 Aug.,	17° 16′ 42″N.	Refr. less parallax,	−30
Cor. for 2^h 48^m,	−1 53	Semi-diam., - -	+15 48
Declination,	17° 14′ 49″N.	True altitude, -	56° 34′ 47″
or - -	17° 14′·82 N.	or - -	56° 34′·78
Polar distance,	72° 45′·18	Z. distance, - -	33° 25′·22

Probable size of error, = ±0′·3

Chronometer fast of apparent time at P(2), -	5^h 21^m 43^s
Loss on apparent time in 3 hours, - - -	−0·5
Gain by sailing west, - - - - - -	+1 6·1
Fast of apparent time at P(3), - - - - -	5^h 22^m $48^s·6$
Chronometer time, - - - - - -	3 49 11
Sun's hour angle, - - - - - - -	1^h 33^m $37^s·6$

Probable size of error in h, ±0′·31; h=23° 24′·4.

To find $A(h, z)$,	log sin (72° 45′·18), - -	9·9800
	log sin (23° 24·4), - -	9·5991
	log cosec (33° 25′·22), - -	0·2590
	log sin $A(h, z)$ (=43° 32′·2), - - - -	9·8381

$$dA(h) = \pm 0'{\cdot}31 \tan (43^\circ\ 32') . \cot (23^\circ\ 24'),$$

$$dA(z) = \pm 0'{\cdot}3 \tan (43^\circ\ 32') . \cot (33^\circ\ 25').$$

The former is evidently the larger, and therefore need alone be calculated.

log 0·31, - - - -	$\bar{1}$·4916
log tan (43° 32′), - - -	9·9778
log cot (23° 24′), - - -	0·3638
log $dA(h)$ (=0·68), - - - -	$\bar{1}$·8332

$\therefore$ $A(h, z)$=S. 43° 32′·2 E., probable size of error ±0′·68.

z= 33° 25′·22	$\therefore$ Σ=75° 55′·4	log cosec, -	0·0123
Δ= 72 45 ·18	$\Sigma-z$=42 30 ·2	log sin, -	9·8297
c= 45 40 ·42	$\Sigma-\Delta$= 3 10 ·2	log cosec, -	11·2572
	$\Sigma-c$=30 15	log sin, -	9·7022
151° 50′·82			0·8023
Σ= 75° 55′·41		log tan (68° 20′·7), - - -	0·4011

$\therefore$ $A(z, c)$ - - =N. 136° 41′·7 E.
=S. 43° 18′·6 E.

Since the probable sizes of the errors of c and z are both ± 0′·3, and $c > z$, $\therefore$ $S > h$, $dA(c) > dA(z)$, therefore calculate the former only.

log 0·3, - - -	0·4771
log cosec (45° 40′), -	0·1455
log cot (23° 24′), - -	0·3638
log 0·97, - - .	$\bar{1}$·9864

$\therefore$ $dA(c) = \pm 0'·97$.

Hence $A(z, c)$=S. 43° 18′·6 E., probable size of error ±0′·97.

To calculate $A(h, c)$, we have

$$\Delta+c=118°\ 25'·9,\ \therefore\ \frac{\Delta+c}{2}=59°\ 12'·8;\ \Delta-c=27°\ \ 4'·8,$$

$$h=23°\ 24'·4,\qquad \therefore\ \frac{h}{2}=11°\ 42'·2;\ \frac{\Delta-c}{2}=13°\ 32'.4.$$

log cot (11° 42′·2), -	0·6837	do., - - -	0·6837
log cos (13° 32′·4), -	9·9878	log sin, - - -	9·3694
log sec (59° 12′·8), -	0·2908	log cosec, - - -	0·0659
log tan (83° 46′·5),	=0·9623	log tan (52° 45′·2),	=0·1199

$\therefore \frac{A+S}{2} = 83° \ 46'{\cdot}4$ $\qquad \therefore \ A(h, c) = $ S. 43° 28′·3 E.

$\frac{A-S}{2} = 52° \ 45'{\cdot}2$ $\qquad$ Probable size of error $= \pm 0'{\cdot}97$.

$A = 136° \ 31'{\cdot}7$
$S = \ \ 31° \ \ 1'{\cdot}3$, $\quad$ since $dA(c)$ is $> dA(h)$.

Hence $\quad A(h, c) = $ S. 43° 28′·3 E., value 1·03,
$\qquad A(z, c) = $ S. 43 18·6 E., value 1·03,
$\qquad A(z, h) = $ S. 43 32·2 E., value 1·47,

$$\therefore A = \text{S. } 43° \ 18'{\cdot}6 \text{ E.} + \frac{9'{\cdot}7 \times 1{\cdot}03 + 13'{\cdot}6 \times 1{\cdot}47}{3{\cdot}53} \text{E.}$$

$$= \text{S. } 43 \ \ 18{\cdot}6 \text{ E.} + 8'{\cdot}5 \text{ E.} = \text{S. } 43° \ 27'{\cdot}1 \text{ E.},$$

Probable size of error $\pm$ 0′·7.

At P(3) the observation for bearing gave A. 97° 12′ Sun's N L.
Sun's semi-diam., 15·8

log cos sun's altitude,		∴	A. 97° 27′·8 ⊙
(56° 34′·8), - -	9·7410	∴	Horiz. angle, A. 103 38 ⊙
log cos (97° 27′·8), -	9·1135		True bearing
			Sun's centre, N. 136 32·9 E.
log horizon. angle, -	9·3725		

From P(3), true bearing A., - - -	N. 32° 54′·9 E.
Inclination of Mercatorial to true meridian, -	+4·7
From A. Mercatorial bearing of P(3), -	S. 32° 59′·6 W.

Draw the straight lines *CD*, *EW* (Fig. 16), cutting each other at right angles in the point *F*; let *CD* represent the meridian of 63° 50′ W. longitude, and *EW* the parallel of 44° 20′ N. latitude; taking half an inch to a mile of latitude in latitude 44° 20′ for the scale of projection, project station A. Mercatorially by its latitude and longitude, drawing its meridian right through the sheet; lay down the meridian of 63° 40′ W. and 64° 0′ W.; draw the meridians and parallels of latitude corresponding to those of P(1), P(2), and P(3) respectively; their points of intersection give the first approximate positions of P(1), P(2), and P(3) on the sheet; from A. draw the straight line AS(1), making with A.'s meridian on

its east side an angle 20° 16′·35, this is the Mercatorial bearing of P(1) from A. given by the observations, and if no error existed would pass through the position of P(1) already projected; but owing to errors it will not generally do so, but will form with the other two straight lines defining the position of P(1) a small triangle, within which the correct position of P(1) will most probably be; draw in a similar manner from A. the Mercatorial bearings of P(2) and P(3), viz. AS(2) and AS(3), which, with the other two lines also defining respectively the positions of P(2) and P(3), will form two other small triangles within which will be the most probable places of P(2) and P(3). To determine the positions of these points within their respective triangles, we proceed as follows. Taking P(1) as an example, we have

Probable size	of the error of	P(1)'s latitude, - -	± 0′·6,
Do.	do.	departure, - -	± 0·3,
Do.	do.	Mercatorial bearing,	± 0·2.

Now A., P(1), is about twenty miles, and therefore the probable displacement of P(1) from the error in the bearing will be about 0′·001, perpendicular to AS(1), and is so small compared with the other errors that it may be neglected. Therefore, taking 0′·6 as difference of latitude, and 0′·3 as departure, we find, using a common traverse table course, 26½° distance 0′·67; through the intersection of P(1)'s meridian with its parallels of latitude, draw the straight line q_1r_1 making with the meridian an angle of 26½°, and cutting AS(1) in P(1), which will give the position of the ship at the first observation as accurately as the scale of projection will admit; in a similar manner draw q_2r_2 cutting AS(2) in P(2), and q_3r_3 cutting AS(3) in P(3). Of course P(1), P(2), and P(3), should be within their respective triangles, but their distances from AS(1), AS(2), and AS(3) respectively are too small to be perceptible; but when the scale of projection and the error in the bearing are sufficiently large, the exact position within the triangle is determined as follows.

Let AB, AC (Fig. 17) be two straight lines intersecting each other at right angles in the point A; AC the meridian, and AB the parallel of latitude defining P(1); draw BC making with the meridian AC the angle $ACB = 20° \ 16'{\cdot}35$; then BC will represent the part of AS(1) intercepted between the meridian and parallel of latitude of P(1); draw AD making with AC the angle $CAD = 26\frac{1}{2}°$, and cutting BC in D; the angle $ADB = 26\frac{1}{2}° + 20° \ 16' = 46° \ 46'$; if p be the correct position of P(1) on AD, we shall have

$$Dp:BA::0'{\cdot}001 \operatorname{cosec}(46^\circ\ 46'):0'{\cdot}67;$$

$$\therefore\quad Dp:DA::14:6714,$$

$$\text{or}\qquad Dp=\frac{14\times DA}{6714}$$

gives p the most probable position of P(1) to be derived from the observations.

In the figure $AD=4{\cdot}6$ inches;

$$\therefore\ Dp=\frac{14\times 4{\cdot}6}{6714}\text{ inches}=0{\cdot}01\text{ inches nearly.}$$

This gives the scale of projection about 6·85 inches to a mile.

We can use the Pole star when visible to determine the true bearing of a light, its deviation from the true north being quickly and easily ascertained, with sufficient accuracy for such purposes, by adding six hours to the sidereal time at ship, and taking from Table 1 in the "Nautical Almanac," for determining the latitude from an observed altitude of Polaris, the correction there given. This will be very nearly the angular distance Polaris is east or west of the meridian, east from sidereal time $1^h\ 9^m$ to sidereal time $13^h\ 9^m$, and west from $13^h\ 9^m$ to $1^h\ 9^m$. From it may be calculated the angle between the meridian and a circle of altitude passing through the Pole star, by adding to the log sine of the correction taken from Table 1 the log secant of the altitude of Polaris; the sum will be the log sine of the true bearing of Polaris. The following example will elucidate this.

On the 7th September 1861, in latitude 44° 5′ N., and longitude $4^h\ 15^m$ west, the observed angle between Polaris and a known light was, when corrected for index error, 87° 2′; the time by a good chronometer $2^h\ 17^m\ 6^s$ fast of Greenwich mean time was $7^h\ 49^m\ 23^s$.

Hence, Greenwich mean time, 6 Sept., -	$17^h\ 32^m\ 17^s$
Longitude west, - - - - - -	4 15
Ship mean time, 6 Sept., - - -	$13^h\ 17^m\ 17^s$
Sidereal time of mean noon at this instant,	11 15 7
Sidereal time at ship, - - - - -	$0^h\ 22^m\ 24^s$
Adding 6 hours, - - - - - -	6 22 24
Cor. from Table 1, "Nautical Almanac,"	
corresponding to $0^h\ 22^m\ 24^s$, Sid. T., -	−1° 24′ 17″
do. do. 6 22 24 „ -	17 0

Latitude, - -	44° 5′	log sin 17′, - -	7·694
Polaris above pole, -	1 24	log sec 45° 29′, -	0·154
Approx. alt. Polaris,	45° 29′	log sin 24′, - -	7·848
True bear. of Polaris,	N. 0° 24′ E.	log cos 37° 2′, - -	8·7139
Hor. angle „	85 46 Lt.	log sec 45° 29′, - -	0·1542
True bear. of light,	N. 86° 10′ E.	log cos 85° 46′, - -	8·8681

When a ship is running in shore, with a line of soundings, and it becomes too dark to see any shore points except lights, Polaris used in this manner becomes a very good third point, and seen in conjunction with two lights, making a sufficient angle, gives a first rate projection. When the Pole star is not available, any bright star whose hour angle is about 6 hours will do exceedingly well. For example, on the 21st October 1863, in latitude 44° 8′ N. and longitude $4^h\ 14^m\ 20^s$ W., the sidereal time at ship being $22^h\ 29^m\ 5^s$, the observed altitude of α Tauri was 11° 29′ after applying the index error of the sextant, and the observed angle between Cross Island light and α Tauri was 95° 21′, no index error, the height of the eye above the sea was 18 feet, with a good horizon. The angle between Iron Bound light and Cross Island light, taken at the same time with a sextant having no index error, was 36° 25′.

Observed alt. α Tauri,	11° 29′	log cos 95° 22′, - -	8·971
Dip and refraction, -	−9	log sec 11 20, - -	0·009
True altitude, - -	11° 20′	log cos 95 29, - -	8·980
Zenith distance, -	78° 40		

$\therefore z = 78°\ 40'$	$\therefore \Sigma - z = 20°\ 29'$	log sin, - -	- 9·5440
$c = 45\ \ 52$	$\Sigma - c = 53\ \ 17$	„ - -	- 9·9039
$\Delta = 73\ \ 46$	$\Sigma - \Delta = 25\ \ 23$	log cosec, - -	- 0·3679
Sum, 198° 18′			
$\Sigma = 99°\ \ 9'$		log cosec, - -	- 0·0056
			19·8214
		log tan 39° 9′, -	- 9·9107

∴ True bearing α Tauri, - - - N. 78° 18′ E.
Horizontal angle Cross Island light, 95 29′ α Tauri.

True bearing Cross Island light,	- N. 17° 11′ W.
Iron Bound light,	36 25 Cr. Is. lt.
True bearing Iron Bound lighthouse,	N. 53° 36′ W.

The Mercatorial bearings of the ship from these two lights, calculated from these in the way already pointed out, will give a good projection of the ship's position.

Sometimes the horizon is not sufficiently distinct to allow a good altitude of a star to be observed; but if the sidereal time at ship can be determined from the known error and rate of a good chronometer, combined with the ship's dead reckoning in the manner before described, the star's hour angle can be estimated sufficiently near the truth to give a good true bearing of the star, provided it is not too near the prime vertical. Take the following example.

On the 10th September 1857, in latitude 43° 33′ N., and longitude 65° 2′ W., with a good sidereal chronometer fast of the ship's sidereal time $7^h\ 4^m\ 25^s$, the following observations, the mean of three of each, gave chronometer time $2^h\ 58^m\ 35^s$, α Boötes Arcturus 54° 21′ Gull Rock light, Shelburne light 41° 12′ Gull Rock light, altitude of α Boötes, 16° 10′ (not good); the height of the eye above the sea 18 feet; the sextant angles being all corrected for index error.

Chronometer, -	$2^h\ 58^m\ 35^s$	log cos 54° 21′, - -	9·7655
Fast of sid. time, -	7 4 25	log sec 16 3, - -	0·0173
Sidereal time, -	$19^h\ 54^m\ 10^s$	log cos 52° 40′, - -	9·7828
Right asc. α Boötes,	14 9 10		
α Boötes hour angle,	$5^h\ 45^m\ 0^s$	Horizontal angle α Boötes,	52° 40′ Gull lt.

$$\begin{cases} c = 46°\ 27 & \therefore \dfrac{\Delta + c}{2} = 58°\ 15'{\cdot}5 \text{ and } \dfrac{h}{2} = 2^h\ 52^m\ 30^s. \\ \Delta = 70°\ 4' & \dfrac{\Delta - c}{2} = 11°\ 48'{\cdot}5. \end{cases}$$

log cot $2^h\ 52^m\ 30^s$, -	0·0284	do. - - -	0·0284
log cos 11° 48′·5, -	9·9907	log sin, - - -	9·3110
log sec 58 15·5, -	0·2789	log cosec, - - -	0·0704
log tan $\frac{A+S}{2}$ (=63°16′·5),	0·2980	log tan $\frac{A-S}{2}$ (=14°24′·5),	9·4098

$$\therefore \quad A = 77°\ 41' \text{ and } S = 48°\ 52';$$

$$\therefore \text{Error in } A = -\text{error in } h \times \frac{\cos 48°\ 52' . \sin 77°\ 41'}{\sin (5^h\ 45^m)}$$

$$= -\text{error in } h \times 0{\cdot}644.$$

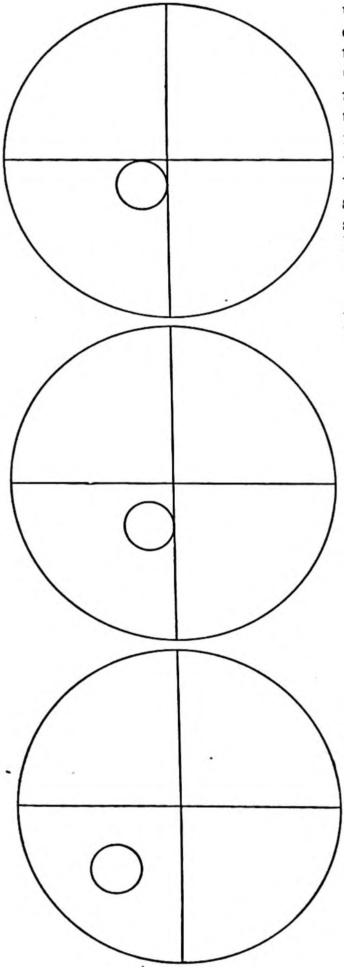
From which it appears that an error of 4^s in the sidereal time will only give an error of $0'\cdot6$ in the true bearing. From this we find

True bearing *a* Boötes, - -	N. 77° 41′ W.
Horizontal angle *a* Boötes, - -	52 40 Gull light.
True bearing of Gull light, - -	N. 25° 1′ W.
Shelburne light, - - - -	41 12 Gull light.
True bearing of Shelburne light, -	N. 66° 13′ W.

We may here observe that in this position of *a* Boötes an error in the latitude of the place, unless very large, will not sensibly affect the true bearing.

From the true bearings thus determined the position of the ship should be roughly projected in order to obtain the Mercatorial correction to the true bearings, and thus determine the Mercatorial bearings of the vessel from the lighthouses, with which project the vessel on the Mercator's sheet; having done this, if the estimated longitude used in finding *a* Boötes' hour angle differs sufficiently from that of the projected position of the ship, the calculations must be corrected, and a more accurate position of the ship obtained.

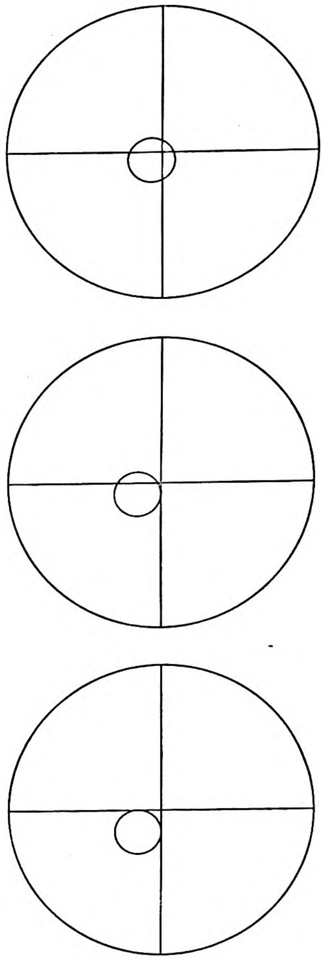
To use a theodolite to determine the direction of the meridian, proceed as follows. Before sunrise, set up the theodolite at the place where the direction of the meridian is required, making the instrumental adjustments with great care, and if the theodolite is not fitted like an altitude and azimuth instrument with a stride level for the horizontal axis of its telescope, apply a scale of equal divisions whose value is known to the cross level on the movable plate, and this must be carefully read before and after every observation whilst the upper plate is clamped on the sun; set the *A* vernier at 360°, read off the others, and note them all; be very particular that the vertical wire of the telescope is in its proper position, direct the telescope to a distant well-defined station, clamp the lower plate firmly, and by means of its tangent screw make the cross of the telescope wires intersect the station; examine the instrument, which, if not in complete adjustment, must be made so, before the observations are commenced, as no adjustment should be made until the observation has been completed, the instrument referred to the zero mark, and all the verniers read and noted. Place on the eyepiece of the telescope a coloured glass suitable to the brightness of the sun, unclamp the



Figs. 17, 18, 19.

upper plate, set the altitude circle to 1°; and supposing the inverting tube of the telescope is used, which is to be preferred, and also that the sun will cross the meridian to the south of the observer, upon turning the telescope towards the sun at rising it will be seen (apparently) above the horizontal wire, and the upper plate must be turned until the image of the sun appears to the right of the vertical wire, as in Fig. 17. Clamp the upper plate, and by means of its tangent screw bring the left limb of the sun, as seen through the telescope, to touch the vertical wire, as in Fig. 18; and keep it in contact by turning the tangent screw slowly and uniformly. During this the sun's motion will cause its image to approach the horizontal wire. When sufficiently near the observer directs his assistant to count the seconds of a mean time chronometer; the instant the sun's limb touches the horizontal wire, the observer ceases turning the tangent screw of the upper plate, and estimates the time of contact from the seconds counted by his assistant. The image of the sun will at the same instant appear as in Fig. 19, the observer reads off the verniers of the upper plate, which together with the time is noted by his assist-

ant. During this time the sun's motion has brought its image into a position like that represented in Fig. 20; the observer turns the tangent screw of the upper plate, so as to bring the vertical wire of the telescope to touch the sun's limb, as in Fig. 21, and keeps it so by turning the screw slowly and uniformly; the sun's motion will make its image appear to move so as to bring its upper limb to touch the horizontal wire, and when it has approached sufficiently, he directs his assistant to count seconds. The instant the sun's limb touches the horizontal wire (as shown in Fig. 22), the observer ceases turning the tangent screw, and estimates the time from the seconds counted by his assistant, which the latter immediately notes in the observation book. He reads off the verniers of the upper plate in succession, the cross level scale, unclamps the telescope, brings the bubble of the telescope level into its central position, and reads off the vertical circle vernier, all of which must be carefully noted by the assistant. Unclamp the upper plate, remove the coloured glass from the eyepiece, and turning the upper plate in the same direction as the sun is moving, bring the cross of the telescope wires to intersect the zero mark, clamp the



FIGS. 20 21, 22.

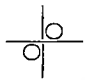
upper plate, and make the intersection exact by using the tangent screw; read off and note the vernier readings; the forenoon observation for this altitude will now be finished. Unclamp the upper plate and the telescope, which set to another altitude 2° or 2° 30′, according to the sun's motion in altitude, so as to give the observer plenty of time to set the altitude correctly, and bring the telescope into the observing position (see Fig. 19), and take a second set in the same manner. He can continue these observations during the forenoon whilst the sun's altitude does not exceed 35°; in the afternoon of the same day, similar observations at the same altitudes must be made, but in reverse order, the largest altitude coming first.

When the altitude of the sun exceeds 15°, and another observer with a sextant and artificial horizon is available, it will be better to observe the equal altitudes of the sun with the sextant, the altitude observer stopping the others; in this case the telescope of the theodolite is set at the given altitude, and it, as well as the index of the sextant, remains unaltered throughout the observation, the observer devoting the whole of his attention to keep the vertical wire in contact with the sun's limb until stopped by the altitude observer.

If the sun's declination was always the same, the half sum of the mean of the readings of the upper plate verniers, corresponding to the same altitude before and after noon, would give the reading of the instrument corresponding to the direction of the meridian passing through the theodolite. The sun, however, has a motion in declination which renders it necessary to apply a correction to the half sum of the before-mentioned readings in order to obtain the reading of the direction of the meridian; this correction is found as follows. Take very accurately from the "Nautical Almanac" the hourly change in the sun's declination at the time of noon at the theodolite on the day of observation, multiply it by the half interval between the A.M. and P.M. observations; to the logarithm of this quantity expressed in seconds of angle add the log cosecant of the half interval, and the log secant of the latitude of the place; the sum will be the logarithm of the correction expressed in seconds of angle which must be applied to the half sum of the readings to obtain the reading corresponding to the direction of the meridian; this correction must be added when the sun is moving *from* the elevated pole, and subtracted when moving *towards* that pole. In making the calculation four places of figures will be sufficient.

The following example will make this clear.

ON THE 29TH JUNE 1854, AT A STATION IN 44° 28′ N. LATITUDE, THE FOLLOWING OBSERVATIONS WERE MADE WITH A 10-INCH THEODOLITE.

Sun's Alt.	Bubble.		Chron. A.M.	Object.	Readings of Theodolite.		
	Alt.	Cross Level.			Vernier A.	B.	C.
		Div.		Inverting Tube Zero station.		120°	240°
	″		h m s		° ′ ″		
					360 0 0	0′ 10″	0′ 5″
5°	+20	0	5 1 24		242 33 30	33 35	33 35
	+30	+1·0	5 4 36		0 35 35	35 45	35 40
				Zero back.	359 59 55	0 10	0

IN THE AFTERNOON OF THE SAME DAY THE OBSERVATION WITH THE SAME INSTRUMENT AND OBSERVER AT THE SAME ALTITUDE GAVE—

Sun's Alt.	Bubble.		Chron. P.M.	Object.	Readings of Theodolite.		
	Alt.	Cross Level.			Vernier A.	B.	C.
		Div.		Same Tube. Zero back.		120°	240°
	″		h m s		° ′ ″		
					360 0 5	0′ 10″	0′ 10″
5°	+40	+4·0	7 21 26		119 10 30	10 35	10 35
	+30	+3·0	7 17 54		0 8 45	8 50	8 50
				Zero station.	360 0 0	0 10	0 5

The plane described by the line of collimation of the telescope, when moved in altitude, was inclined to the axis about which the horizontal plate turned at a constant angle of 50″, and a division of the cross level scale was 13″ of angle.

In considering this observation we have for the

A.M. observation, telescope altitude bubble reading, -	+25″
P.M. do. do. do., -	+35″
Difference, - - - - - - - - - -	10″

From which it appears that the altitude of the sun at the P.M. observation was 10″ less than at the A.M. observation, and in consequence the reading of the horizontal plate at the P.M. observation was larger than it should have been. Examining the observations we immediately perceive that whilst the sun moved through its diameter in altitude, it moved in bearing 1′ 45″ less than its diameter. The sun's diameter, as given by the "Nautical Almanac," was 31′ 32″; hence the 10″ too little altitude in the P.M. observation gave 9″ too much bearing reading; and therefore half this, or 4″·5, must be subtracted from the half sum of the A.M. and P.M. bearing readings of the theodolite. From the readings of the cross level we have

Mean of A.M. readings, - -	+0·5	scale divisions.
„ P.M. „ - -	+2·0	„
„ A.M. and P.M. readings,	+2·0	„

Hence	Permanent level error, - -	+50″
	Cross level, 2 divisions, or -	26″
	Total level error, - -	76″

by which the east end of the axis, about which the telescope of the theodolite turned in altitude, was too high when the telescope pointed south. To determine the effect of this on the bearing of the south line as given by the theodolite—to the logarithm of the level error in seconds add the log tangent of the sun's altitude, the sum will be the logarithm of the error in the bearing expressed also in seconds.

Hence	log 76″, - - -	1·8808
	log tan (5°),- - -	8·9419
	log correction (6″·6), -	0·8227

To determine the correction for the sun's change of declination in the interval between the A.M. and P.M. observations, the "Nautical Almanac" gives the sun's hourly motion in declination at noon station time 8″·6.

The mean of the A.M. times was	- -	5^h	3^m	0^s
„ „ P.M. „	- -	19	19	40
Interval between A.M. and P.M. observations,		14^h	16^m	40^s
Half interval, -	- - - - -	7^h	8^m	20^s

∴ Change in sun's declination for half interval = 61″·4.

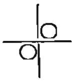
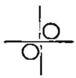
To the logarithm of this quantity add the log cosecant of the half interval, and the log secant of the latitude—the sum will be the logarithm of the correction expressed in seconds to be applied to the theodolite reading of the bearing; in the present case to be added, because the sun was moving south, and the latitude of the place was north :

log 61·4, - - - -	1·7882
log cosec (7^h 8^m 20^s), - -	0·0196
log sec (44° 28′), - - -	1·1465
log correction (90″), - -	1·9543

Arithmetic mean of the zero readings,	360° 0′ 5″·4
„ mean of the bearing, -	361 44 17 ·5
Difference, - - - - - -	1° 44′ 12
Half difference, or reading of south, -	52′ 6
Correc. for sun's motion in declination,	+1 30
„ level error of instrument, -	+ 6 ·6
„ difference in setting the alt.,	− 4 ·6
Corrected reading of South, - -	0° 53′ 38″
True bearing of the zero mark, - -	S. 0° 53′ 38″ E.

On the same day the following observations were taken after the sun was sufficiently high to enable its altitude to be observed with a sextant and artificial horizon; the telescope of the theodolite was set to half the sextant reading, and was not altered in altitude during the observation. The theodolite observer can in this way devote his whole attention to keep the vertical wire of the telescope in contact with the sun's limb until stopped by the sextant observer, who gives him notice of the near approach of the contact of the sun's limbs, when he desires the assistant with the chronometer to count

seconds; in the table subjoined the mean of the three vernier readings of the upper plate are given.

2 Sun's Altitude by Sextant.	Cross Level Bubble.	Time A.M.	Object.	Mean of the three Vernier Readings.	Cross Level Bubble.	Time P.M.	Object.	Mean of the three Vernier Readings.
	Div.	h	Zero stat.	° ′ ″	Div.	h	Zero back.	° ′ ″
$\overline{\odot}$	−1·0	6		360 0 3	+1·5	5		360 0 0
°		m s 23 41		256 7 10		m s 58 47		105 37 50
37		26 45		0 3 10		55 44		0 41 46
$\underline{\odot}$	−1·5		Zero back.	359 59 55	+1·0		Zero stat.	360 0 5
		h	Zero stat.	° ′ ″		h	Zero back.	° ′ ″
$\overline{\odot}$	−2·0	7		360 0 0	+0·5	4		360 0 0
°		m s 43 12		269 10 0		m s 39 15		92 34 45
65		46 13		0 3 8		36 15		0 41 52
$\underline{\odot}$	−1·5		Zero back.	359 59 50	+1·0		Zero stat.	360 0 4

We have for altitude 18° 30′

Cross level reading A.M. mean,	- -	−1·25 Div.
„ „ P.M. „	- -	+1·25
„ „ mean of A.M. and P.M.,		0

Permanent level error 50″, log =	-	1·6990
log tan (18° 30′), -	-	9·5245
log correction (16″·7),	-	1·2235

Mean of A.M. time,	$6^h\ 25^m\ 13^s$	Change of sun's dec. in $5^h\ 46^m = 49''\cdot6$, log -	1·6955
„ P.M. „	17 57 15	log sec (44° 28′), -	0·1465
Interval, - -	$11^h\ 32^m\ 2^s$	log cosec (5° 46′ 1″), -	0·0008
Half interval, -	$5^h\ 46^m\ 1^s$	log correction (69″·6),	1·8428

Arithmetic mean of zero readings, - -	360° 0′ 0″·8
„ „ bearing, - - -	361 45 8
Difference, - - - - - -	1° 45′ 7″·2
Half do., or theodolite reading of South, -	52′ 33″·6
Correction for sun's change in declination, -	+1′ 9 ·6
„ level error, - - - -	+ 16″·7
True bearing of zero station, - - -	S. 0° 54′ 0″ E.

For alt. 32° 30′ we have cross level reading A.M. mean,			−1·75 div.
Do.	do.	P.M. „	+0·75 „
Cross level mean A.M. and P.M., -			−0·5

∴ Cross level error, -	−6″·5		
Permanent, - -	50 ·0		
Total level error, -	43″·5	log 43·5, - - -	1·6385
		log tan (32° 30′), -	9·8052
		log correction (27″·8),	1·4437

Mean A.M. times,	7^h 44^m $42·5^s$	Change in sun's dec. in	
Mean P.M. times,	16 37 45	4^h 26^m 31^s = 38″·2, log,	1·5821
		log cosec (4^h 26^m 31),	0·0372
Interval, - -	8^h 53^m $2·5^s$	log sec (44° 28′), -	0·1465
Half interval, -	4^h 26^m 31^s	log correction (58″·3),	1·7658

Arithmetic mean of zero readings, -	359° 59′ 58″·5
Do. bearing, - -	361 44 52 ·5
Difference, - - - - - -	1° 44′ 54″
Half do., - - - - - - -	0° 52′ 27″
Cor. for sun's change of declination, -	+58 ·3
„ level error, - - - -	+27 ·8
True bearing of zero station, - -	S. 0° 53′ 53″ E.

To compare the relative value of these three observations, we must determine the probable size of their respective errors.

In the first observation the probable size of the error in setting the telescope altitude was considered = ±15″; the pro-

bable size of the error in making the contact between the vertical wire of the telescope and the sun's limb $= \pm 12''$; the probable size of the error made in determining the level error $= \pm 10''$; also $A = 61° 41'$; $S = 43° 8'$ and $h = 7^h 8^m 20^s$.

Hence probable size of the error in the bearing arising from errors in setting the telescope altitude

$$= \pm 15'' \times \cos(43° 8') . \operatorname{cosec}(45° 32') . \operatorname{cosec}(7^h 8^m 20^s);$$

∴ log 15, - - -	1·1761
log cos (43° 8′), -	9·8631
log cosec (45° 32′), -	0·1465
„ ($7^h 8^m 20^s$),	0·0196
log (16″), - -	1·2053

The probable size of the error in the bearing from the error in making the contact between the vertical wire of the telescope and the sun's limb $= \pm 12''$. sec sun's alt. $= \pm 12''$. sec 5°:

log 12, - - -	1·0792
log sec (5°), - -	0·0017
log (12″), - -	1·0809

Probable size of the error in the bearing from the error in determining the level error $= \pm 10''$. tan 5°:

log 10, - - -	1·0000
log tan (5°), - -	8·9419
log (0″·9), - -	$\bar{1}$·9419

The probable size of the error in the true bearing arising from these three errors combined is the same as that of the largest $= \pm 16''$.

In the second and third sets the error in determining the level error was the same size as in the first; but in setting the sextant the probable size of the error in the altitude was considered to be $= \pm 7''$; and that in making the contact between the vertical wire of the telescope and the sun's limb $= \pm 8''$.

In No. 2 we have $A = 76° 11'$; $S = 48° 58'$; $h = 5^h 46^m 1^s$,

In No. 3 we have $A = 88° 13'$; $S = 50° 56'$; $h = 4^h 26^m 31^s$.

In No. 2, sun's altitude was 18° 30′, which gave

log 7, - -	0·8451	log 8, - -	0·9031	log 10, - -	1·0000
log cos (48° 58′),	9·8172	log sec (18° 30′),	0·0230	log tan (18° 30′),	9·5245
log cosec (45° 32′),	0·1465	log error (8″·4),	0·9261	log error, - -	0·5245
log cosec ($5^h 46^m 1^s$),	0·0008				
log error, - -	0·8096				

Hence, 8″·4 being the largest of the three, the probable size of the error of No. 2 bearing is ± 8″·4.

In No. 3 observation the sun's altitude is 32° 30′, which gives

log 7, - -	0·8451	log 8, - -	0·9031	log 10, - -	1·0000
log cos (50° 56′),	9·7995	log sec (32° 30′),	0·0740	log tan (32° 30′),	9·7042
log cosec (45° 32′),	0·1465	log error (9″·5),	0·9771	log error, -	0·8042
log cosec($4^h 26^m 31^s$)	0·0372				
log error, - -	0·8283				

Therefore the probable size of the error in No. 3 bearing is ± 9″·5.

Valuing each of these in the inverse proportion of the probable sizes of their respective errors, if we assume the value of No. 1 to be 6·25, we shall have the value of No. 2 = 11·9, and that of No. 3 = 10·5.

∴ No. 1, true bear. of zero station	S. 0° 53′ 38″ E.,	value	6·25
No. 2, „ „	0 54 0	„	11·9
No. 3, „ „	0 53 53	„	10·5
			28·65

$$\text{Mean} = \text{S. } 0° \ 53' \ 38'' \text{ E.} + \frac{22'' \times 11{\cdot}9 + 15'' \times 10{\cdot}5}{28{\cdot}65}$$

$$= \text{S. } 0° \ 53' \ 38'' + 15'' \text{ E} = \text{S. } 0° \ 53' \ 53'' \text{ E.},$$

∴ True bearing of zero station from the observing station = S. 0° 53′ 53″ E.

CHAPTER VI.

BASE LINES.

THE means within the reach of a nautical surveyor for measuring the lengths of lines he wishes to determine in feet and its fractions, though sufficient for practical purposes, when properly understood and used, are at best rough and inaccurate; and therefore a mile of latitude at the middle latitude of the survey is the best unit of length to be adopted in such surveys, because it can be determined with considerable accuracy from the astronomical differences of latitude carefully observed at the extreme north and south limits of the survey. It has besides the advantage of being the unit which navigators, for whom the chart is constructed, must necessarily use. Base lines must be measured, nevertheless, wherever the nature of the country affords a sufficient length of level ground, because it is necessary to find the relation which the nautical mile adopted for the unit of linear measure bears to the foot given by the brass scale usually supplied to surveying vessels.

To enable a surveyor to measure a base line, we shall suppose that he has an ordinary four-foot brass scale, a good steel chain 100 feet long, properly adjusted to a given temperature, and its expansion due to 1° Fahrenheit carefully noted in his instrument book.

The steel chain must be compared with the brass scale before commencing to measure a base line; whilst doing so the chain and scale must be kept in the shade, and as nearly as possible at the same temperature. The comparison is made in the following manner.

Prepare battens about four or five inches broad and two inches thick, and tack them along the deck, fore and aft, so that they may extend evenly about 104 feet in that direction, testing the upper surface with the longest steel straight-edge on board; strike a straight line fore and aft along the middle of the upper surface of the battens; with a pair of beam compasses take off four feet accurately from the brass scale,

commencing at a fixed point on the central straight line about two feet from the after end of the battens, measure forward with the beam compasses on the *left* side of the straight line, but as close to it as possible, 100 feet, making the 25th mark with the foremost point of the beam compass on the central line instead of close to its side; the distance between this point and the starting point should be 100 feet.

In order to test the accuracy of the measurement thus made, remeasure back again, still keeping the front or aftermost point of the beam compass on the left of the central straight line when looking *aft*, which will now be on the opposite side of the line from that on which the measure was made in the former case: this will prevent the point of the beam compass from being disturbed, by falling into one of the point holes made in the wood by the beam compass leg whilst measuring forward. The last mark made by the beam compass which completes the hundred feet of the return measure must be made on the central line, and ought to coincide with, or be very near to, the starting point of the measurement; otherwise the process must be continued until the agreement between the two measures is sufficiently near. During the process of measuring, the beam compass must be frequently applied to the brass scale, and the exact distance between its two points noted.

The arithmetic mean of the measures, after they have been corrected for temperature and beam compass errors, is taken, and 100 feet accurately set off on the central straight line. The steel chain, which should have been previously stretched along the deck by the side of the battens, must now be laid upon them, stretched tightly along the central straight line, and the difference between its length and the 100 feet just measured accurately ascertained and noted, as well as the temperature; and after a base line has been measured, the chain should be tested in the same way and its error ascertained.

Where a good level piece of ground of sufficient length can be found, measure a base line as follows. Level a smooth straight path in the direction of the longest piece of level ground, so as to have a clear view of the whole line from each of its extremities, which should be so situated as to command objects from which the triangulation can be carried on in such a situation. In the middle of one end of the base line path thus constructed, set up a vertical pole A, and at the middle point of the path at its other extremity drive a peg B into the ground, over this peg set up a theodolite, so that its vertical axis may pass through the centre of the peg B; clamp its

upper plate, and set the telescope of the theodolite so that its vertical wire coincides with the vertical pole *A*. In the mean time having previously prepared three lengths of deep sea line, each 100 fathoms long, between two marks placed about 2 fathoms from each of its ends, secure the end of one of these lines to the foot of the pole *A*, so that the mark may lie at the extremity of the diameter of a horizontal section of the pole, which is perpendicular to the base line. On the left side when looking towards *B*, stretch the line tightly towards *B*, and abreast the mark near the other end of the deep sea line; and on its right side, looking towards *B*, place a vertical pole about half *A*'s length securely in the ground, so that it may coincide exactly with the vertical wire of the telescope of the theodolite at *B*, with the top of *A* seen over it, also coinciding with the vertical wire; call this base pole No. 1. It will be on the base line about 100 fathoms from *A* towards *B*. Secure the other end of the first line to this pole, the 100 fathom mark being on its left side; secure the mark at one end of the second deep sea line to the foot of No. 1 pole, so as to lie exactly under or over the mark of the first line, and stretch it along the middle of the base path tightly towards *B*, and at the mark at its other end place a second pole in a similar manner, which call No. 2 base pole, which will be distant from *A* about 200 fathoms towards *B*. In the same way with the third deep sea line, place base pole No. 3, which will be about 300 fathoms from *A* towards *B*. It will be convenient to station two men to each deep sea line, one at each end, to attend, secure, and shift it. The base poles, when once satisfactorily and firmly placed on the base line, must remain untouched, and will be found, after the base line has been satisfactorily measured, useful fixed points of reference when putting in the topography in their vicinity, their distances from *A*, as given by the chain, having been carefully determined and noted. As soon as the first deep sea line has been secured to *A* and the base pole No. 1, it must be carefully struck straight between them; after which the chain party can commence measuring. Starting from *A*, and stretching the chain tightly close along the left side of the deep sea line towards No. 1 pole, the sixth arrow of the first tally should be close to it; the second arrow of the second tally should be close to No. 2 pole, and so on. The distances of the base poles from *A* must be ascertained and noted; besides being useful as before mentioned, they serve to check the tallies, and prevent large errors. After the chain measurers have passed one chain length beyond No. 1 pole towards No. 2, the first line must be shifted so that the end that was attached to *A* may be secured

in the same manner to base pole No. 3, and the fourth pole placed on the base line in the same manner as the other three; and when firmly planted the end of the line formerly secured to No. 1 pole must now be secured to it, after the line has been hauled sufficiently tight. When all the poles have

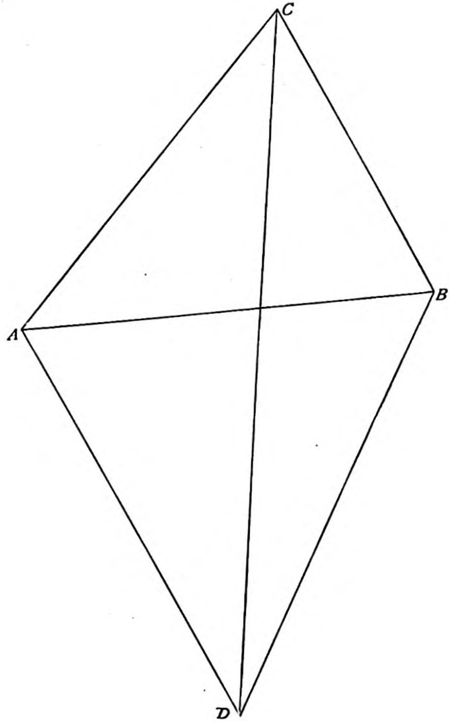


FIG. 23.

been placed on the base line, the base line measured, and the theodolite angles observed at *B*, the theodolite may be taken down, and a pole similar to that at *A* must be erected where the peg *B* was driven.

After the chain measurement from *A* to *B* has been com-

pleted, the deep sea lines must be shifted to the other side of the poles, and commencing at B, the base must be measured from B to A, keeping the chain close to the left side of the lines when looking from B to A. After the back measurement is finished, and both the measurements corrected for temperature and chain errors, the two distances must be compared, and if they agree or do not differ much from each other, the arithmetic mean of the two must be taken as the correct length of the base line from A to B; but if the difference is not sufficiently small the measurements must be repeated. The chain should be kept shaded, and as nearly at the same temperature as possible.

Let AB (Fig. 23) be the base line, whose length has been measured in the manner just described; C and D stations in commanding positions, one on each side of AB, selected for the purpose of carrying on the triangulation, both of which are seen from A and B and from each other respectively, and such that the angles ACD and ADB each exceed 30°.

The same observer with the same theodolite having with equal precision observed the angles CAB, BAD, DBA, ABC, BCD, DCA, ADC, and CDB—

Let n_1 be the sum of the errors of the three observed angles of the triangle ACD,

n_2,	- -	BCD,
n_3,	- -	ABC,
n_4,	- -	ABD,

Let also $2n_1x$ be the error made in observing the angle CAD:

	$2n_2y$, - - -	DBC,
	$2n_3z$, - - -	BCA,
	$2n_4w$, - - -	ADB,
	n_1x+p, - - -	CAB.
Therefore	n_1x-p, - - -	BAD.
Assume	n_2y+p, - - -	ABC,
∴	n_2y-p, - - -	DBA.
Let	n_3z+q, - - -	BCD,
∴	n_3z-q, - - -	DCA.
Assume	n_4w+q, - - -	CDB,
∴	n_4w-q, - - -	ADC.

Hence

$$\left.\begin{aligned} n_1x+p+n_2y+p+2n_3z&=n_3,\\ n_1x-p+n_2y-p+2n_4w&=n_4,\\ n_3z+q+n_4w+q+2n_2y&=n_2,\\ n_3z-q+n_4w-q+2n_1x&=n_1.\end{aligned}\right\}\quad\text{(A)}$$

Adding the first two equations in (A),

$$2n_1x+2n_2y+2(n_3z+n_4w)=n_3+n_4.$$

Taking their difference,

$$4p+2(n_3z-n_4w)=n_3-n_4.$$

Assume $x=y$ and $z=w$ as being probable, and we have

$$2(n_1+n_2)x+2(n_3+n_4)z=n_3+n_4;$$

but $$n_1+n_2=n_3+n_4,$$

and consequently $$x+z=\tfrac{1}{2}, \quad \ldots\ldots(1)$$

$$p=\frac{n_3-n_4}{4}(1-2z), \quad \ldots\ldots(2)$$

In like manner $$q=\frac{n_2-n_1}{4}(1-2x). \quad \ldots\ldots(3)$$

Having yet another probable assumption to make, let $x=z$, and therefore from equation (1), $x=\frac{1}{4}$, $z=\frac{1}{4}$, and similarly $y=\frac{1}{4}$ and $w=\frac{1}{4}$. Substituting z by its value thus determined in equation (2), we have

$$p=\frac{n_3-n_4}{8}.$$

Similarly, $$q=\frac{n_2-n_1}{8};$$

$\therefore$ error in angle $CAB=n_1x+p=\dfrac{n_1}{4}+\dfrac{n_3-n_4}{8}$,

" " $BAD=n_1x-p=\dfrac{n_1}{4}+\dfrac{n_4-n_3}{8}$,

" " $ABC=n_2y+p=\dfrac{n_2}{4}+\dfrac{n_3-n_4}{8}$,

" " $DBA=n_2y-p=\dfrac{n_2}{4}+\dfrac{n_4-n_3}{8}$,

" " $BCD=n_3z+q=\dfrac{n_3}{4}+\dfrac{n_2-n_1}{8}$,

" " $DCA=n_3z-q=\dfrac{n_3}{4}+\dfrac{n_1-n_2}{8}$,

" " $CDB=n_4w+q=\dfrac{n_4}{4}+\dfrac{n_2-n_1}{8}$,

" " $ADC=n_4w-q=\dfrac{n_4}{4}+\dfrac{n_1-n_2}{8}$,

These corrections being applied to the observed angles, we shall obtain the most probable value of the angles of the quadrilateral figure $ACBD$. Take the following example:—

$AB=56813{\cdot}4$ scale feet.

$$
\begin{array}{llrrrr}
\text{Observed angles} & CAB = & 48^\circ & 38' & 25'' \\
\quad\text{,,} \quad\quad \text{,,} & BAD = & 66 & 12 & 12 \\
\quad\text{,,} \quad\quad \text{,,} & DBA = & 63 & 24 & 40 \\
\quad\text{,,} \quad\quad \text{,,} & ABC = & 67 & 24 & 47 \\
\quad\text{,,} \quad\quad \text{,,} & BCD = & 29 & 16 & 11 \\
\quad\text{,,} \quad\quad \text{,,} & DCA = & 34 & 40 & 36 \\
\quad\text{,,} \quad\quad \text{,,} & ADC = & 30 & 28 & 44 \\
\quad\text{,,} \quad\quad \text{,,} & CDB = & 19 & 54 & 28 \\
\hline
\text{Sum} \quad - \quad - \quad - & = & 360^\circ & 0' & 3'' \\
\end{array}
$$

$$\therefore \quad n_1 + n_2 = n_3 + n_4 = 3''.$$

$$
\text{Triangle } ACD \left\{ \begin{array}{rrrr}
\text{angle } A = & 114^\circ & 50' & 37'' \\
C = & 34 & 40 & 36 \\
D = & 30 & 28 & 44
\end{array} \right.
$$

$$\text{Sum} \quad - \quad - \quad - \quad = 179^\circ \; 59' \; 57'' \qquad \therefore n_1 = \; -3''$$

$$
\text{Triangle } DBC \left\{ \begin{array}{rrrr}
\text{angle } B = & 130^\circ & 49' & 27'' \\
C = & 29 & 16 & 11 \\
D = & 19 & 54 & 28
\end{array} \right.
$$

$$\text{Sum} \quad - \quad - \quad - \quad = 180^\circ \; 0' \; 6'' \qquad \therefore n_2 = 6''$$

$$
\text{Triangle } ABC \left\{ \begin{array}{rrrr}
\text{angle } A = & 48^\circ & 38' & 25'' \\
B = & 67 & 24 & 47 \\
C = & 63 & 56 & 47
\end{array} \right.
$$

$$\text{Sum} \quad - \quad - \quad - \quad = 179^\circ \; 59' \; 59'' \qquad \therefore n_3 = 1''$$

$$
\text{Triangle } ABD \left\{ \begin{array}{rrrr}
\text{angle } A = & 66^\circ & 12' & 12'' \\
B = & 63 & 24 & 40 \\
D = & 50 & 23 & 12
\end{array} \right.
$$

$$\text{Sum} \quad - \quad - \quad - \quad = 180^\circ \; 0' \; 4'' \qquad \therefore n_4 = 4''$$

Error in angle $CAB = -\frac{3}{4}'' - \frac{5}{8}'' = -\frac{11}{8}'' = -1''{\cdot}4$,

$\therefore$ correction $= 1''{\cdot}4$,

and angle CAB corrected $= 48^\circ \; 38' \; 25'' + 1''{\cdot}4$

$= 48 \;\; 38 \;\; 26 \; {\cdot}4$,

$$
\begin{array}{lll}
\text{error angle } BAD = -\tfrac{3}{4}'' + \tfrac{5}{8}'' = -\tfrac{1}{8}'' = -0''\cdot 1; & \therefore \text{ cor.} & = +0''\cdot 1 \\
\quad ,, \qquad ABC = \tfrac{6}{4}'' - \tfrac{5}{8}'' = \tfrac{7}{8}'' = 0''\cdot 9, & \quad ,, & = -0\cdot 9 \\
\quad ,, \qquad DBA = \tfrac{9}{4}'' + \tfrac{5}{8}'' = \tfrac{17}{8}'' = 2''\cdot 1, & \quad ,, & = -2\cdot 1 \\
\quad ,, \qquad BCD = -\tfrac{1}{4}'' + \tfrac{9}{8}'' = \tfrac{7}{8}'' = 0''\cdot 9, & \quad ,, & = -0\cdot 9 \\
\quad ,, \qquad DCA = -\tfrac{1}{4}'' - \tfrac{9}{8}'' = -\tfrac{11}{8}'' = -1''\cdot 4, & \quad ,, & = +1\cdot 4 \\
\quad ,, \qquad CDB = 1'' + \tfrac{9}{8}'' = 2''\cdot 1, & \quad ,, & = -2\cdot 1 \\
\quad ,, \qquad ADC = 1'' - \tfrac{9}{8}'' = -0''\cdot 1, & \quad ,, & = 0\cdot 1
\end{array}
$$

$$
\begin{array}{llll}
\therefore \text{ angle } DCA = & 34^\circ\ 40'\ 36'' + 1''\cdot 4 = & 34^\circ\ 40'\ 37''\cdot 4 \\
\quad ,, \qquad ADC = & 30\ \ 28\ \ 44\ + 0\cdot 1 = & 30\ \ 28\ \ 44\cdot 1 \\
\quad ,, \qquad CAD = & 114\ \ 50\ \ 37\ + 1\cdot 5 = & 114\ \ 50\ \ 38\cdot 5 \\
\hline
\text{Sum} \quad - \quad - \quad - & & 180^\circ\ \ 0'\ \ 0''
\end{array}
$$

The angles of the triangle BCD become when corrected as follows:

$$
\begin{array}{lll}
\text{angle } DBC = & 130^\circ\ 49'\ 27'' - 3'' & = 130^\circ\ 49'\ 24'' \\
\quad ,, \qquad BCD = & 29\ \ 16\ \ 11\ - 0\cdot 9 & = 29\ \ 16\ \ 10\cdot 1 \\
\quad ,, \qquad CDB = & 19\ \ 54\ \ 28\ - 2\cdot 1 & = 19\ \ 54\ \ 29\cdot 9 \\
\hline
\text{Sum of the angles} \quad - & & 180^\circ\ \ 0'\ \ 0''
\end{array}
$$

Triangle ABC gives

$$
\begin{array}{lll}
\text{angle } BCA = & 63^\circ\ 56'\ 47'' + 0''\cdot 5 = & 63^\circ\ 56'\ 47''\cdot 5 \\
\quad ,, \qquad CAB = & 48\ \ 38\ \ 25\ + 1\cdot 4 = & 48\ \ 38\ \ 26\cdot 4 \\
\quad ,, \qquad ABC = & 67\ \ 27\ \ 47\ - 0\cdot 9 = & 67\ \ 24\ \ 46\cdot 1
\end{array}
$$

Sum of the corrected angles of $ABC = 180^\circ\ \ 0'\ \ 0''$

Triangle ABD gives

$$
\begin{array}{lll}
\text{angle } ADB = & 50^\circ\ 23'\ 12'' - 2'' & = 50^\circ\ 23'\ 10'' \\
\quad ,, \qquad ABD = & 66\ \ 12\ \ 12\ + 0\cdot 1 & = 66\ \ 12\ \ 12\cdot 1 \\
\quad ,, \qquad DBA = & 63\ \ 24\ \ 40\ - 2\cdot 1 & = 63\ \ 24\ \ 37\cdot 9
\end{array}
$$

Sum of the corrected angles of $ABD = 180^\circ\ \ 0'\ \ 0''$

From the triangle ABC we calculate BC and AC thus—

log AB(=56813·4),	4·7544508	- - - -	4·7544508
log sin(48°38′26″·4),	9·8753971	log sin(67°24′46″·1),	9·9653410
log cos(63 56 47 ·5),	0·0465377	- - - -	0·0465377
log BC(=47466·3),	4·6763856	log AC(=58388·8),	4·7663295

From triangle ABD

log AB, - -	4·7544508	- - -	4·7544508
log sin (66°12′12″·1),	9·9614132	log sin (63°24′37″·9),	9·9514524
log cosec(50°23′10″),	0·1133070	- - -	0·1133070
log BD(= 67479·7),	4·8291710	log AD(= 65949·3),	4·8192102

Since the angles at C and D are all smaller than 45°, CD can be determined more accurately by using the cosines of the angles. Since

$$CD = AC \cos ACD + AD \cos ADC$$
$$= BC \cos BCD + BD \cos CDB,$$

log AC, - -	4·7663295	log AD, - -	4·8192102
log cos (34°40′37″·4),	9·9150683	log cos (30°28′44″·1),	9·9354145
log 48017·31, -	4·6813978	log 56836·15, -	4·7546247

$\therefore$ $CD = 104853·46$ scale feet.

log BC, - -	4·6763856	log BD, - -	4·8291710
log cos(29°16′10″·1),	9·9406810	log cos (19°54′25″·9),	9·9732412
log 41406·3	4·6170666	log 63447·16 -	4·8024122
63447·16			

$\therefore$ $CD = 104853·46$, which agrees exactly with the former value and shows that the geometrical relations of the quadrilateral figure $ACBD$ have been exactly satisfied, and the measured base line AB has thus been satisfactorily extended to CD.

The length of the vessel's mast may be used to determine the distance between two stations on shore; but the method of chaining a base line just described is far preferable, and must be always adopted when possible.

Measure with great care and accuracy the distance that the main or highest truck is *vertically* above a fixed mark in the ship's side, a short distance above the water, when she is perfectly upright; two boards, ten or twelve feet long, painted a colour that contrasts most distinctly with that of the vessel's side, and nailed along the sides of the vessel a short distance above

the water, to the surface of which their edges are parallel and equidistant, and so that the vertical plane passing through the truck, and perpendicular to the line of keel, may bisect the edges of the boards respectively on each side of the vessel, will answer the purpose exceedingly well.

Let A (Fig. 24) be the vessel's truck, B the fixed point vertically below it, and a short distance above the surface

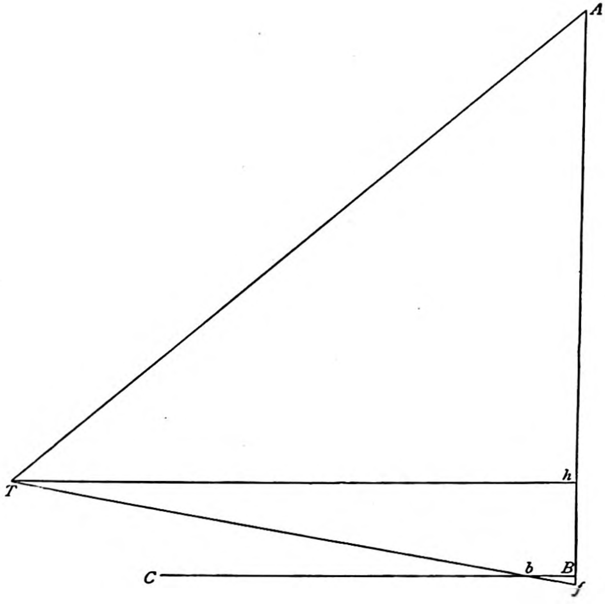


FIG. 24.

of the sea, to which the straight line BbC passing through B, and b the middle point of the upper edge of the board nailed on the ship's side in the manner before recommended, is parallel; also, let T be a theodolite on shore in the vertical plane passing through A, B, and b (we may here observe that when the ship's hull is black the boards should be painted white);

join the points A and B, and also T and b, by straight lines; produce these to meet in the point f, join TA, and through T draw Th parallel to CbB and meeting AB in h.

Let $AB=h$, and $Bb=b$, $\alpha=$the reading of the vertical circle of the theodolite when the cross of its wires intersects A, the reading when it intersects b being denoted by γ, and β the reading when the cross of the wires would intersect h, or when the telescope bubble has its central position.

When Th is very much longer than AB, which is generally the case, and when T is not at a great elevation above the sea, the angle TfA will be so nearly equal to a right angle that it may be taken as such without introducing any error of practical importance. Under this consideration we have

$$\begin{aligned} Th &= Af \cot ATb, \\ Bf &= Bb \tan Bbf, \\ &= Bb \tan hTb; \end{aligned}$$

$$\begin{aligned} \therefore\ Th &= (AB+Bf) \cot ATb, \\ &= AB \cot ATb + Bb \frac{\tan hTb}{\tan ATb}, \\ &= h \cot(\alpha-\gamma) + b \frac{\tan(\beta-\gamma)}{\tan(\alpha-\gamma)} \ldots\ldots\ldots\ldots\ldots\ldots (1) \end{aligned}$$

If the vertical plane passing through the theodolite and the truck does not also pass through the point b, as we have supposed, but is inclined to the line of the vessel's keel at an angle θ, we have only to replace b in equation (1) by $b \sin\theta$, and the general expression for the value of Th will therefore be given by

$$Th = h \cot(\alpha-\gamma) + b \sin\theta \frac{\tan(\beta-\gamma)}{\tan(\alpha-\gamma)} \ldots\ldots\ldots\ldots (2)$$

from which it appears that the smaller $(\beta-\gamma)$ and the smaller θ are, the better will be the position of the theodolite.

Anchor, or rather moor the vessel in the middle of a bay, so as to be as nearly as possible equidistant between two theodolite stations which can see each other and the vessel, so that the straight line joining the two stations may pass just clear of the ship, and with the angles of elevation of the truck above the edge of the boards not differing much from two degrees. From these stations at appointed times the angles of elevation of the ship's truck above the edge of the board must be *simultaneously* observed, the time of observation being defined by dipping a ball and flag from the truck.

Let T_1 (Fig. 25) be one of the theodolite stations, and the plane of the paper the horizontal plane passing through it, T_2 the projection of the other theodolite station on the plane of the paper by a vertical straight line passing through it, A that of the ship's truck, and AS the intersection of the vertical plane passing through the line of the ship's keel with the plane of the paper.

Let the angle $AT_1T_2 = T_1$,
Do. $T_1T_2A = T_2$,
Do. $SAT_1 = \theta_1$,
Do. $T_2AS = \theta_2$.

α_1, β_1, and γ_1 the readings of the vertical circle of the theodolite at T_1, and α_2, β_2, and γ_2 those observed at T_2; then

$$AT_1 = h\cot(\alpha_1 - \gamma_1) + b\sin\theta_1 \frac{\tan(\beta_1 - \gamma_1)}{\tan(\alpha_1 - \gamma_1)}, \quad (3)$$

$$AT_2 = h\cot(\alpha_2 - \gamma_2) + b\sin\theta_2 \frac{\tan(\beta_2 - \gamma_2)}{\tan(\alpha_2 - \gamma_2)}, \quad (4)$$

$$T_1T_2 = AT_1\cos T_1 + AT_2\cos T_2, \ldots\ldots\ldots\ldots (5)$$

From which the horizontal distance between T_1 and T_2 is found.

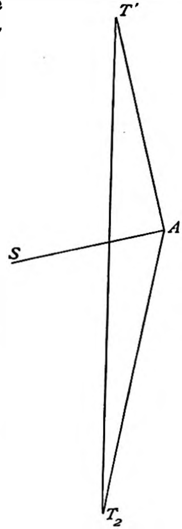


FIG. 25.

In the following example the observations were taken simultaneously at T_1 and T_2, the time being defined by signals from the ship, after the instruments at T_1 and T_2 were adjusted and their A verniers of the upper plate set to 360°. A distant well-defined zero mark was selected for each, and during the observations the A verniers only were read and noted; particular and careful attention was paid by each observer to reading the vertical circles of their respective theodolites. The time for making each signal was arranged beforehand, and the corresponding time of each chronometer noted in the observer's book, so as to inform him when to expect them.

The crosses of the theodolite wires were kept intersecting the truck by the tangent screws of the upper plates and vertical circles, which were relinquished the instant the dip was observed, and the chronometer times taken and noted; the vertical circles of the theodolite were then read with great care, and by means of the tangent screws of the vertical circles the crosses of the wires of the telescopes were made to coin-

cide with the middle point of the upper edge of the board in view of the theodolite, corresponding to the point *b* in Fig. 25, and the vertical circle read and noted; the bubble of each telescope was then brought to its central position and the vertical circle read and noted; the *A* vernier of the horizontal plate was next read. This completed the first observation of the set.

The observer at each theodolite then unclamped the upper plate and brought the cross wire of his theodolite to coincide with the vessel's truck ready for the second signal, and in this way the following observations were taken.

OBSERVATIONS MADE AT T_1 THEODOLITE.

Time by Chronometer.	Object.	A Reading.	Vertical Circle.
		° ′ ″	° ′ ″
Dip 1.	Zero mark, -	360 0 0	
	Truck, - -	45 15 20	1 41 30
h m s 8 30 5	Water line mark, centre of board edge,	- - -	−12 15
	Bubble, -	- - -	− 0 45
	T_2 - -	50 2 10	
Dip 2.	Truck, - -	45 20 15	1 40 45
h m s 8 35 10	Water line mark, centre of board edge,	- - -	−12 15
	Bubble, -	- - -	− 0 50
Dip 3.	Truck, - -	45 18 10	1 41 5
h m s 8 40 1	Water line mark, centre of upper edge,	- - -	−12 10
	Bubble, -	- - -	− 0 45
Dip 4.	Truck, - -	45 22 40	1 40 30
h m s 8 45 4	Water line mark, centre of upper edge of board,	- - -	−12 10
	Bubble, -	- - -	− 0 55
Dip 5.	Truck, - -	45 17 50	1 41 15
	Water line mark, -	- - -	−12 15
h m s 8 50 1	Bubble, -	- - -	− 0 50
	T_2 - -	50 2 0	
	Zero back, -	360 0 0	

OBSERVATIONS MADE AT T_2 THEODOLITE.

Chronometer Time.	Object.	A Reading.	Vertical Circle.
		° ′ ″	
Dip 1.	Zero mark, -	360 0 0	° ′ ″
	Truck, - -	82 34 50	1 32 5
h m s	Water line mark, centre edge of board,	- - -	−25 10
8 31 0	Bubble, -	- - -	+ 0 20
	T_1, - -	77 39 20	
Dip 2.	Truck, - -	82 34 25	1 32 55
h m s	Water line mark, -	- - -	−25 20
8 36 4	Bubble, -	- - -	+ 0 15
Dip 3.	Truck, - -	82 34 35	1 32 30
h m s	Water line mark, -	- - -	−25 15
8 40 55	Bubble, - -	- - -	+ 0 15
Dip 4.	Truck, - -	82 33 55	1 33 10
h m s	Water line mark, -	- - -	−25 10
8 45 59	Bubble, -	- - -	+ 0 20
Dip 5.	Truck, - -	82 33 30	1 32 15
	Water line mark, -	- - -	−25 10
h m s	Bubble, -	- - -	+ 0 25
8 50 56	T_1, - -	77 39 30	
	Zero back, -	360 0 15	

The angles observed at the ship with a sextant corrected for index error were

At Dip 1, angle T_1AS	88° 2′,	At Dip 1, angle AST_2	82° 16′
„ 2, „	90 13,	„ 2, „	80 10,
„ 3, „	88 51,	„ 3, „	81 30,
„ 4, „	89 56,	„ 4, „	81 45,
„ 5, „	88 37,	„ 5, „	81 14,
Arithmetic mean θ_1=	89 8,	Arithmetic mean θ_2=	81 14.

Half the breadth of the ship at B was 15 feet.

The height of the vessel's truck above the upper edges of the boards was 127·5 feet.

The arithmetic mean of the observations at T_1 gives

$$\begin{array}{lll} \text{Mean value of} & \alpha_1 = & 1^\circ\ 41'\ \ 1'', \\ \text{Do.} & \beta_1 = & -49\ , \\ \text{Do.} & \gamma_1 = & -12\ \ 13\ , \end{array}$$

$$\therefore\ \alpha_1-\gamma_1 = 1^\circ\ 53'\ 14'';\ \beta_1-\gamma_1 = 11'\ 24'',$$
$$\theta_1 = 89^\circ\ \ 8';\ T_1 = 4^\circ\ 43'\ 16''.$$

The arithmetic mean of the observations at T_2 gives

$$\begin{array}{llll} \text{Mean value of } \alpha_2 = 1^\circ\ 32'\ 35'' & \therefore\ \alpha_2-\gamma_2 = 1^\circ\ 57'\ 50'', \\ \text{Do.}\quad \beta_2 = \quad +19 & \beta_2-\gamma_2 = \quad 25\ \ 34\ , \\ \text{Do.}\quad \gamma_2 = \ -25\ \ 15, & \end{array}$$

$$\theta_2 = 81^\circ\ 14', \text{ and } T_2 = 4^\circ\ 54'\ 50''.$$

Referring to equations (3), (4), and (5)—
from (3), $AT_1 =$
$127{\cdot}5 \cot(1^\circ\ 53'\ 14'') + 15 \sin(89^\circ\ 8') \tan(11'\ 24'') \cot(1^\circ\ 53'\ 14'')$,

log 15, - - -	1·176091
log sec 89° 8′, - -	9·999950
log tan 11′ 24″, -	7·520633
log 0·05, - - -	8·696674

$\therefore\ AT_1 (=127{\cdot}5+0{\cdot}05) \cot(1^\circ\ 53'\ 14'')$ feet.

log 127·55, - -	2·105680
log cot (1° 53′ 14″), -	1·482143
log AT_1(=3871 feet),	3·587823
log cos T_1(=4° 43′ 16″),	9·998524
log(=3857·86 feet), -	3·586347

From (4),

$AT_2 = \{127{\cdot}5 + 15 \sin(81^\circ\ 14').\tan(25^\circ\ 34')\} \cot(1^\circ\ 57'\ 50'')$,

log 15, - - .	1·176091	log 127·61, - -	2·105885
log sin (81° 14′), ..	9·994896	log cot (1° 57′ 50″),	11·464836
log tan (25° 34′),	7·871408		
		log AT_2, - -	3·570721
log 0·11, - ..	$\bar{1}$·042395	log cos (4° 54′ 50″),	$\bar{9}$·998401
127·5		log (3707·84 feet), -	$\bar{3}$·569122
127·61		3857·86 „	

$\therefore\ T_1T_2 = 7565{\cdot}7$ feet.

It is obvious that a very small error in AB will make a considerable error in the distance T_1T_2, which shows how very important it is that AB should be very accurately measured. This is difficult to do mechanically; but if the vessel is in any harbour which has been well surveyed, and two stations such as T_1 and T_2 can be found, where the distance T_1T_2 can be taken accurately from the chart, then by reversing the process we can determine AB from T_1T_2 with great accuracy, and the distance AB thus ascertained can be afterwards used to measure a distance in the way described.

The interval between the flash and report of a gun fired at a given place may be used to determine its distance from an

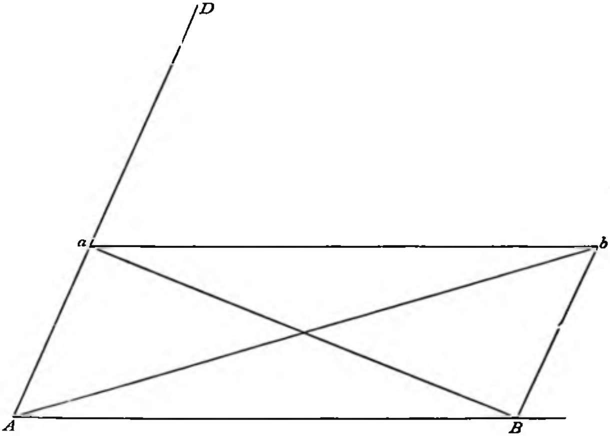


FIG. 26.

observer furnished with a good second-hand watch, who must be very careful to observe and note the exact time of seeing the flash and that of his hearing the report; because an error of a tenth of a second of time in the interval will give an error of about 110 feet in the distance to be determined; the effect of the wind, though small in proportion to that of an error in the interval, must be taken into account and eliminated by placing a gun at each of the stations, the distance between which is to be found in this way, and firing them alternately the same number of times: the stations ought to be a long distance from each other, and if it is intended to resort to this method, observations should be made beforehand between two

stations as far apart as possible, of which the distance is accurately known; the rate and direction of the wind, and the temperature of the air being observed at the same time by this means, the velocity of sound and the effect on it produced by temperature and wind may be determined and tabulated for future use. This method can only be resorted to in exploring voyages, when the other methods before described are not available.

Let A and B be the two stations from which the guns are to be fired; DA the direction of the wind.

The mean of the time intervals observed at A between the flash and report of the gun fired at B was t_1 seconds, the mean of those observed at B was t_2 seconds; let v_s be the velocity of sound expressed in feet per second of time, and v_w that of the wind.

From B draw Bb parallel to AD, and make $Bb=t_1v_w$; and in AD take $Aa=t_2v_w$; join Ab and Ba by straight lines.

When the report of the gun fired at B is heard at A, the centre of the disturbance due to its explosion will have reached b, and bA will be equal to t_1v_s; in like manner aB is equal to t_2v_s.

The triangle ABb gives $Ab \sin BAb = Bb \sin ABb$,

$$\therefore \sin BAb = \frac{v_w}{v_s} \sin ABb.$$

Similarly $$ABa = \frac{v_w}{v_s} \sin BAa.$$

$\because$ Bb is parallel to AD, the angle $ABb = BAa$,

and $\therefore$ angle BAb = angle ABa;

since, v_w being much less than v_s, both these angles must be less than right angles.

Let $BAD = W$ and $BAb = ABa = \alpha$,

$$\therefore AB = aB \cos \alpha + Aa \cos W,$$

also $$AB = Ab \cos \alpha - Bb \cos W,$$

$$\therefore 2\,AB = (aB + Ab)\cos \alpha + (Aa - Bb)\cos W$$
$$= (t_2 + t_1)v_s \cos \alpha + (t_2 - t_1)v_w \cos W \quad\text{.........(1)}$$

By subtracting the above expressions for AB,

$$0 = (t_2 - t_1)v_s \cos \alpha + (t_2 + t_1)v_w \cos W \quad\text{...........(2)}$$

also $$\sin \alpha = \frac{v_w}{v_s} \sin W \quad\text{...................................(3)}$$

Combining (2) and (3),

$$\tan \alpha = \frac{t_1 - t_2}{t_1 + t_2} \tan W \quad\text{......................(4)}$$

from which, when W is not too near a right angle, the value of a can be found; when v_w is known, equation (3) will give the value of a.

Eliminating cos W between (1) and (2), we have

$$2\,AB = v_s \cos a\left\{t_2 + t_1 - \frac{(t_2 - t_1)^2}{t_2 + t_1}\right\},$$

$$\therefore AB = 2\,v_s \cos a \frac{t_1 t_2}{t_1 + t_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

from which when v_s and a are known AB can be calculated.

When the wind is light and nearly in the direction of AB, the angle a will be very small, and we may in such cases take $AB = 2v_s\frac{t_1 t_2}{t_1 + t_2}$ without sensible error.

If AB is known, v_s can be calculated from equation (7), from which

$$v_s = \frac{AB(t_1 + t_2)}{2 \cos a \times t_1 t_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

When used in this way the strength and direction of the wind should be measured, and the barometer and thermometer noted; a being determined by putting an approximate value of v_s in equation (4) when W is nearly 90°.

Take the following example, where $t_1 = 27^s{\cdot}25$, $t_2 = 26^s{\cdot}72$, and $W = 40°$; substituting in equation (4), $\tan a = \frac{53}{5397}\tan 40°$; here three places of figures in the logarithms are quite sufficient.

log 53, - - - -	1·724
log tan 40°, - - -	9·924
AC log 5397, - - -	6·268
log tan a(= 28′ 20″), - -	7·916

v_s was considered to be 1095 feet per second.

$$\therefore\ AB = 2 \times 1095 \frac{27{\cdot}25 \times 26{\cdot}72}{53{\cdot}97} \cos(28'\ 20''),$$

log cos (28′ 20″), - -	9·999985
log 2190, - - -	3·340444
log 27·25, - - -	1·435366
log 26·72, - - -	1·426836
AC log 53·97, - -	8·267848
log AB(= 29544 feet), -	4·470479

From equation (3), $v_w = v_s \sin \alpha \,.\, \operatorname{cosec} W$
$= 1095 \sin(28' \, 20'') \operatorname{cosec} 40°$,

log 1095, - - - -	3·039
log sin (28′ 20″), - -	7·916
log cosec 40°, - - -	0·192
log v_w(= 14 feet per second),	1·147

∴ the velocity of the wind was about $9\frac{1}{2}$ miles an hour.

If, however, AB had been given, and $= 29544$ feet, we should by inverting the process have found $v_s = 1095$ feet per second.

In these observations the same observers using the same instruments should always be employed.

CHAPTER VII.

TRIANGULATION.

THE triangulation of a survey is most important; the positions of the main or principal stations from which the angles are to be observed must be carefully selected. To do this the country must be explored by a well-instructed surveyor, who should be furnished with a small theodolite, angle and sketch books, with which to visit every elevated peak and other commanding place, from which good views of the surrounding country can be obtained, and then pick out the best site for a station.

When the surveyor has arrived at a place of this description, he should first look for two main stations, subtending at his eye an angle not exceeding 150°, and not less than 30°; these stations are to be used to determine the position of the station he is about to erect, and they should be nearer to him than any of the other main stations. Having found these, he should examine the vicinity, keeping the two main stations in sight, in order to pick out the position which, besides commanding the positions where it will be necessary to place subsequent stations, has the greatest command over the near objects surrounding it. In placing a main station, it should be as nearly equidistant as possible from the main stations which are to be used to determine its position, and the angle they subtend at it should be as nearly 60° as possible—the limits being from 90° to 30°. Having settled the position of the station, the surveyor should set up the small theodolite, with its vertical axis passing through the centre of the intended station, and take a round of angles, the *A* vernier readings only being noted—in the round all the main stations, remarkable peaks and other objects worthy of notice should be cut off.

A panoramic view of the surrounding country should be sketched in the book provided for the purpose, marking by means of vertical fine lines passing through them the objects whose angles have been taken with the theodolite, and writing the reading along their upper extremities above the sketch. These sketches will be found very useful, and the angles will enable the principal stations and other points to be projected

on a small scale with a protractor, by which the surveyor can see how they will lie on the paper with respect to each other, and how the sheets for the main projection should be arranged.

The surveyor should always have present to his mind, when selecting the position of a station, the effect which errors of observation and projection will have on the accuracy of its determination The following will assist him in comparing the value of different situations, and estimating the probable sizes of the error of projection and calculation to which their determinations are liable.

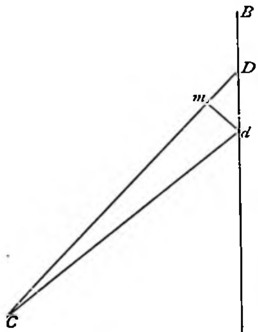


FIG. 27.

Let AB (Fig. 27) be a given straight line passing through D the correct position of a station, C a given point from which the station D was cut off; but from an error in the observation, combined with an error in projection, the straight line drawn from C cuts AB in the point d; join DC, and from centre C describe through d a circular arc dm cutting DC in m.

Let $DCd = a$, the error of an angle made in the observation and projection of Cd, supposed small; then $dm = a \times Cd$ may be considered as a straight line cutting DC at right angles in m; also, let $CDA = \theta$.

Hence, Dd the error in D's projection consequent on the error a

$$= dm \,.\, \text{cosec}\ \theta,$$
$$= a \times Cd \ \text{cosec}\ \theta;$$

consequently, when a and θ are constant, Dd will vary with CD; and the shorter CD is, the smaller this error will be; when a and CD are constant, Dd will vary with cosec θ, and will have its smallest value when $\theta = 90°$. If the value of the error in this case, when $\theta = 90°$, be unity, and the error and distance remain the same—

if θ be changed to		30°,	error changes to		2,
,,	,,	19·75,	,,	,,	3,
,,	,,	14·5,	,,	,,	4,
,,	,,	11·5,	,,	,,	5,
,,	,,	9·5,	,,	,,	6,
,,	,,	8·1,	,,	,,	7,
,,	,,	7·2,	,,	,,	8,

which shows the rapid rate at which the error increases as the cutting angle diminishes after it is less than 30°; this is very important and should be kept constantly in view.

Next, let A and B (Fig. 28) be two known stations, and P the correct position of the station to be determined from angles observed at A, B, and P.

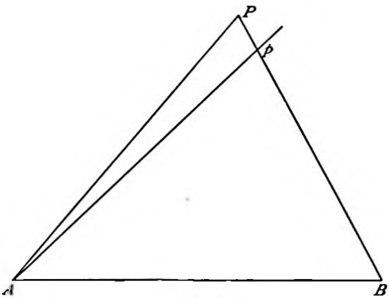


Fig. 28.

Join AP and BP by straight lines, and by means of the angle A, determined from the observations, project the straight line Ap cutting BP in p; then PAp is the error in the angle A depending on the errors of observation and projection; let this $=a$, and the error in P's place consequent on this error will be

$$Pp = a\,AP\,.\,\text{cosec}\,P,$$
$$= a\frac{AB\,.\,\sin B}{\sin^2 P}.$$

Since this error in the angle A may be either in excess or defect, each of which is equally probable, the probable error in P's place from an error in the angle A

$$= \pm a\frac{AB\,.\,\sin B}{\sin^2 P}.$$

In like manner the probable error in P's place from an error in the angle B $\qquad = \pm a\dfrac{AB\,.\,\sin A}{\sin^2 P}$.

The position of P has to be determined by the intersection of the straight lines drawn respectively from A and B, and therefore the whole error in P's place from the two errors combined will be the *algebraic* sum of the above two errors, viz., the sum or difference of their arithmetic values according as they have like or unlike signs. It is evident that the two errors being perfectly independent of each other the probability of their signs being alike is equal to the probability of their signs being unlike, and consequently the most probable size of the combined errors, or the whole error in P's place, will be equal to the half sum of the arithmetical values of the two

errors added to the half sum of their difference, or in other words—the probable size of the error from the combined effects of the two errors will be the same as that of the larger of the two errors taken arithmetically. From this it appears that the probable size of the error in P's place from the errors in the directions of the straight lines drawn from A and B respectively for its projection will be equal to the larger of the two quantities $a\frac{AB \, . \sin A}{\sin^2 P}$, $a\frac{AB \, . \sin B}{\sin^2 P}$, and will therefore have its smallest value when $A = B$ and $P = 90°$; and the nearer these conditions can be obtained, other things being the same, the more accurate will be the projection of P.

Sometimes a very good position for a main station is so situated, with respect to the two established main stations upon which its projection depends, that the angle they subtend at it is too small to afford a satisfactory projection. In this case it is necessary to find another less commanding position, where a station that can be well determined from the two known stations, may be so placed as to give the best possible projection for the main station that we wish to determine.

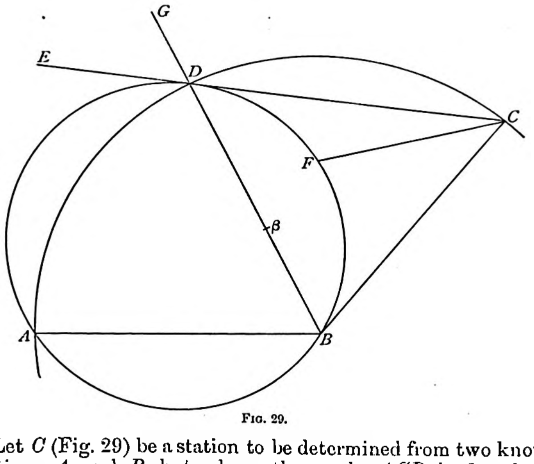


Fig. 29.

Let C (Fig. 29) be a station to be determined from two known stations A and B, but where the angle ACB is found too small to afford a good projection—to find another position D that can be well determined from A and B, in the best possible position for the projection of C.

From the point C draw the straight line CDE so that the angle BCD shall not exceed a right angle; describe a circle ABD, of which the circumference passes through A and B, and touches the straight line CDE, and let D be the point of contact; in the circumference BD take any point F, and join FC, then the angle FCB will be less than the angle DCB, and since DCB is less than a right angle it will be nearer to a right angle than FCB; and therefore DC will cut BC at a better angle than FC, or than any other straight line which can be drawn from the circumference of the circle $BFDA$.

Because the straight line CDE touches the circle $BFDA$, the angle BDC is equal to the angle BAD in the alternate segment; consequently when D is in its best position, the angle BDC is equal to the angle BAD.

Again, as the angle ADB increases to a right angle, the segment $ADFB$ approaches a semicircle, and when it becomes so the circle of which it is a part is the smallest that can pass through the points A and B; consequently as D increases to a right angle the point D approaches B and the angle DCB diminishes; but the nearer these two angles simultaneously approach a right angle the better will be the projection of D, and also of C: this will evidently be when the angle ADB $=$ angle BCD, in which case also the angle $ABD =$ angle DBC.

Because the straight line CDC touches the circle $ABFD$ the angle $ADE =$ angle ABD in the alternate segment $=$ half the angle ABC; therefore angle $ADC = 2$ right angles $- ADE$ $= 2$ right angles $-$ half ABC.

Therefore to determine D draw the straight line BDG bisecting the angle ABC, and find the point D on this line where the angle ADC is equal to $180° - \frac{ABC}{2}$; this is practically done by setting up the small theodolite at B with its zero at A, unclamp the upper plate and bring the cross of the telescope wires on C, clamp the upper plate, make the coincidence of the cross wires with C exact by means of the tangent screw—read off the A verniers and note it—divide the reading by two, unclamp the upper plate, and set the A vernier to the half-reading thus found. Look through the telescope to see if there is any distinct well-defined object coinciding with the cross of the wires, and note it and mark it, so as to know it again, otherwise a mark such as a pole with a flag must be set up there. Suppose this to be the point β, the observer proceeds from B beyond β, then moves on keeping β in one with B, having his sextant set to the angle $180° -$ half the reading of the A vernier. When the reflected image of A coincides with C seen directly, β and B being in one, he will be at the

point D. This point cannot generally be arrived at exactly in practice, as it may be in an obscure place, in which case the observer selects a position from which A, B, and C can be seen as near to the exact position of D as possible.

Otherwise, since $BC = AB\dfrac{\sin BAD . \sin BDC}{\sin ADB . \sin BCD}$,

let $BAD = A$, $ADB = D_1$, $BDC = D_2$ and $BCD = C$.

Suppose an error		α	be made in the angle		A,
Do.	do.	δ_1	do.	do.	D_1,
Do.	do.	δ_2	do.	do.	D_2,
Do.	do.	γ	do.	do.	C.

Then consequent error in BC will be

$$BC \times \{\alpha \cot A + \delta_2 \cot D_2 - \delta_1 \cot D_1 - \gamma \cot C\} \dots\dots\dots (1)$$

Now, as A increases, it is evident that D_2 diminishes; consequently α and δ_2 have opposite signs. In like manner δ_1 and γ must have opposite signs, and in the same triangle there is no reason for supposing that the errors in the angles are unequal in size; and therefore we may suppose as most probable that $\alpha = -\delta_2$ and $\delta_1 = -\gamma$. Substituting these conditions in (1), we have error in BC

$$= BC\{\alpha(\cot A - \cot D_2) + \gamma(\cot D_1 - \cot C)\},$$

which has its smallest possible value when $D_2 = A$ and $D_1 = C$. This is the same condition as that before arrived at by purely geometrical considerations.

When the station cannot be made exactly at the point D, a position not far from it will be practically as good, as the error in BC, being a minimum, increases very slowly in moving from D in any direction; therefore the vicinity of D should be carefully examined before deciding on the position of the station.

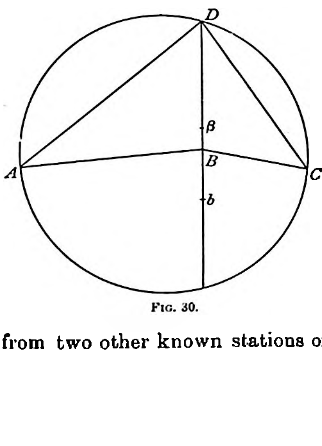


FIG. 30.

This is the way to find the position at which a vessel should be anchored off shore, for the purpose of determining the position of a station on the sea-shore from two other known stations on the same shore to the right

or left of it, as the case may be. I have found the method so useful that a fuller description of the process will be advantageous.

Let A and B (Fig. 30) be two known stations in sight of each other, C a station on the coast line which can see B and also be seen from it. The nature of the country inshore is such that no place can be found suitable for a station to connect C with A and B, in such a situation that its position may be determined with sufficient accuracy, but each of the three stations, A, B, and C, has a clear view to seaward.

Lay the vessel off shore at the point D in sight of the three stations A, B, and C, on the straight line $bB\beta D$, where the angle $ABD=DBC$, and the angle $ADC=180°-ABD$, or as near that position as possible. This can easily be done as follows. By means of the theodolite set up at B place a mark β, which can be seen from the ship on the line BD, and as near to the sea shore as possible, and another mark b inshore of the station on the straight line DB produced, and as far inshore as possible, so that it may be seen over B; run the vessel off shore, keeping β, B, and b in one; if β is not very close to B, it will generally be sufficient to keep the vessel near enough to the straight line BD, without the assistance of b. Besides, by means of signals, the observer at B, when he sees the vessel deviating from her course, can bring her back to BD. With a sextant set to the angle $180°-ABD$, the observer on board the vessel looking at C directly will see the reflected image of A approaching C; when they coincide the vessel is in her place.

If the length of BC can be approximated to sufficiently well, and the vessel lose the line Bb, the angles ADB and BDC can be approximated to, and this should be done before starting as follows.

Let $\quad BC=m^2AB$ approximately,

$$D=90°-\frac{ABD}{2}, \text{ and } ADB=D+\theta.$$

Therefore $$BDC=D-\theta,$$

but $$\frac{\sin^2 BDC}{\sin^2 ADB}=\frac{BC}{AB}=m^2,$$

$\therefore$ $$\sin(D-\theta)=m\sin(D+\theta),$$

or, $$\tan\theta=\frac{1-m}{1+m}\tan D, \qquad (2)$$

from which when m can be approximated to sufficiently well, θ can be found, and the angles ADB and BDC will point out the position of the ship near enough to the exact place of D for all practical purposes.

We may here observe that the observation at B is the most important, as any error in the direction BD will have a greater effect on the position of C than an error of the same size in either of the angles at D or C, and therefore the best observer and the best theodolite should be placed at B; the angles observed with the sextant at D will be useful only in placing the ship.

To make the observation—observers with theodolites are stationed at A, B, and C, having their chronometers compared with that of the ship, and the times at which the signals are to be made noted in their angle books; a ball over a flag makes a good signal; about five minutes before the first observation is to be taken the ball and flag should be hoisted half mast, and the hoisting part of halyards secured; this serves as a preparative; about 15 or 20 seconds before the time for the first signal arrives, hoist the ball and flag close up to the truck, keeping the halyards in hand with the part still secured in the dipping position; at "the instant for giving the signal," at the order *Dip*, the halyards are let go, and the ball and flag drop suddenly into their dipping position; each of the observers keeps the cross of his telescope wires intersecting the truck to which the ball and flag are hoisted, by means of the tangent screw of the horizontal plate, which the observer ceases turning the instant he sees the dip. He takes the time and reads off the verniers of the horizontal plate, which he carefully notes in his angle book. Five or six such observations repeated at intervals of five minutes or thereabouts will suffice to give a very accurate position of C; the ship had better be anchored, if possible, but this is not absolutely necessary.

The following affords a good example of this method of determining the position of an important station on the coast line. In this, AB is 31795 feet, the angle ABC looking seaward was $191° 25' 32''$, and the angle ABD was $95° 42' 46''$; the straight line BD on which the ship was to anchor was defined by a mark β placed to seaward of B as near the high water line as possible, and another mark b was placed on the straight line DB produced inshore, so as to be visible over the station B; the vessel ran off shore, and was anchored nearly on the straight line BD with the angle $CDA = 84° 22'$ nearly. The times the dips were made were taken, and noted in the ship's deck angle book by an observer on board the vessel thus

Dip 1, ship's mean time,		11^h	0^m	2^s
„ 2,	„ „	11	5	1
„ 3,	„ „	11	10	3
„ 4,	„ „	11	15	1
„ 5,	„ „	11	20	0

The observer at *A* set the zero of his theodolite at the station *B*, and observed the ship's truck as follows:—

THEODOLITE OBSERVATIONS AT STATION A.

Object.	Chronometer Time.			A Vernier.			B Vernier.		
	h	m	s	°	′	″	°	′	″
Zero laid at B, -	-	-	-	360	0	0	180	0	45
Dip 1, ship's fore truck,	10	45	41	316	45	10	0	46	0
Zero back, - -	-	-	-	360	0	10	0	0	50
Dip 2, ship's fore truck,	10	59	41	316	43	25	0	44	15
Zero back, - -	-	-	-	360	0	5	0	0	50
Dip 3, ship's fore truck,	11	4	41·5	316	44	30	0	45	10
Zero back, - -	-	-	-	360	0	0	0	0	40
Dip 4, ship's fore truck,	11	9	40	316	45	50	0	46	45
Zero back, - -	-	-	-	359	59	53	0	0	40
Dip 5, ship's fore truck,	11	14	40	316	46	30	0	47	0
Zero back, - -	-	-	-	360	0	0	180	0	40

THEODOLITE OBSERVATIONS AT STATION B.

Object.	Chronometer Time.			A Vernier.			B Vernier.		
	h	m	s	°	′	″	°	′	″
Zero laid at station C,	-	-	-	360	0	0	179	59	30
Dip 1, ship's fore truck,	11	7	18·5	264	19	30	0	19	10
Zero back, - -	-	-	-	359	59	50	0	59	25
Dip 2, ship's fore truck,	11	12	16	264	17	50	0	17	20
Zero back, - -	-	-	-	359	59	55	0	59	30
Dip 3, ship's fore truck,	11	17	20	264	19	0	0	18	20
Zero back, - -	-	-	-	360	0	0	179	59	40
Dip 4, ship's fore truck,	11	22	17	264	20	0	0	19	30
Zero back, - -	-	-	-	360	0	0	0	59	30
Dip 5, ship's fore truck,	11	27	16	264	21	0	0	20	40
Zero back, - -	-	-	-	357	57	55	179	59	30

THEODOLITE OBSERVATIONS AT STATION C.

Object.	Chronometer Time.			A Vernier.			B Vernier.		
	h	m	s	°	′	″	°	′	″
Zero set at station B,	-	-	-	360	0	0	180	1	10
Dip 1, ship's fore truck,	11	10	8	40	56	50	0	58	5
Zero back, - -	-	-	-	360	0	10	0	1	15
Dip 2, ship's fore truck,	11	15	6	40	57	0	0	58	5
Zero back, - -	-	-	-	360	0	6	0	1	10
Dip 3, ship's fore truck,	11	20	9	40	57	0	0	58	10
Zero back, - -	-	-	-	359	59	55	0	1	5
Dip 4, ship's fore truck,	11	25	6	40	56	30	0	57	45
Zero back, - -	-	-	-	360	0	0	0	1	5
Dip 5, ship's fore truck,	11	30	6	40	56	45	0	58	0
Zero back, - -	-	-	-	360	0	0	180	1	10

FROM THESE OBSERVATIONS THE RESULTS IN THE FOLLOWING TABLE WERE OBTAINED.

	Dip 1.	2.	3.	4.	5.
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
Angle BAD =	43 14 51	43 16 39	43 15 34	43 14 0	43 13 34
ABD =	95 45 11	95 43 27	95 44 26	96 45 30	95 46 38
(e) ADB =	40 59 58	40 59 54	41 0 0	41 0 30	40 59 48
Angle BCD =	40 56 48	40 56 52	40 57 1	40 56 36	40 56 48
DBC =	95 40 21	95 42 5	95 42 5	95 40 2	95 38 54
(e) BDC =	43 22 51	43 21 2	43 21 53	43 23 22	43 24 18
Log 31795, -	4·5023588	4·5023588	4·5023588	4·5023588	4·5023588
„ cosec ABD,	0·1830619	0·1830716	0·1830571	0·1829844	0·1830862
„ sin BAD,	9·8357864	9·8360281	9·8358827	9·8356722	9·8356140
„ cosec BCD,	0·1835226	0·1835129	0·1834910	0·1835517	0·1835226
„ sin BDC,	9·8368584	9·8366176	9·8367291	9·8369274	9·8370521
„ BC, - -	4·5415881	4·5415890	4·5415187	4·5414945	4·5416337
∴ BC=34800·7 feet.		34800·8	34795·1	34793·2	34805·4

The arithmetic mean of which gives $BC = 34799{\cdot}1$ feet, with probable error of one observation $= \pm 3{\cdot}87$ feet, and the probable error of the arithmetic mean of all five is $\pm 0{\cdot}77$ feet.

If a ridge of land intervenes between two stations B and C so as to prevent these being mutually seen from each other, but where both B and C are seen from a known station A, and the position of B with respect to A has been accurately determined; to determine C with respect to A and B, a station

O must be made on the ridge, from which A, B, and C can be seen, where the angles BAO, OAC may be as nearly equal

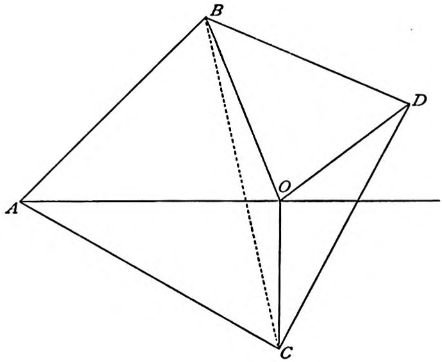


FIG. 31.

as possible, and where the angle BOC may be as nearly equal to $180° - \frac{BAC}{2}$ as possible.

The angles BAO, OAC, OBA, ACO, COA, and AOB must be well and carefully observed; the three angles of the triangle AOB, that is, $BAO + OBA + AOB$, must equal 180°, and this ought to be the case with the sum of the observed angles or very nearly so; if this is not the case the angles must be re-observed; and when the difference between 180° and the sum of the observed angles of the triangle AOB is sufficiently small, one-third of the difference must be applied to each of the angles; the same remark applies to the triangle AOC. When the angles of these triangles have been corrected as above, AO and BO are calculated from AB, and the three angles of the triangle AOB in the usual manner, and afterwards from AO and the angles of the triangle AOC, AC and CO are determined in the same way.

We observe that the two angles AOB, AOC are only connected by their common side AO, and therefore the only check we have on the observed angles is the condition that the three angles taken together of each of the two triangles must separately be equal to 180°.

To calculate BC the side and ABC, ACB the angles of the triangle ABC, there are two ways. First, the two sides AB, AC, and the included angle BAC are known, from which the

other elements of the triangle ABC can be calculated in the ordinary manner; secondly, we have the two sides BO, OC, and their included angle COB; to determine BC and the angles OBC and BCO, since the angle BAC ought not to exceed 60°, each of the angles BAO and OAC ought not to differ much from 30°; consequently neither of the angles OBC nor BCO can differ much from 15°; and therefore generally the best way to calculate BC is from the equation

$$BC = BO \cos OBC + CO \cos BCO, \qquad \text{.................}(3)$$

The angles ABC and ACB immediately result by applying the angles OBC, BCO respectively to the observed angles ABO and ACO; from these angles AC and BC must again be calculated, and ought to agree with their values previously determined.

If it is necessary to use the station C to carry on the triangulation, we must seek for the position of a station D, which can be seen from B and C, and can see them also, and form with them a well-shaped triangle. D should, if possible, see and be seen from O. Suppose this to be the case, and that the angles at A, B, C, D, and O have been equally well observed, in order to determine the most probable value of the corrections that should be applied to the observed angles to make them consistent with each other, proceed as follows. Referring to Fig. 31,

Let the small error of observation in the angle $BAO = w_1$,
that of the angle $BAC = w_2$,
" " $DBO = x_1$,
" " $DBA = x_2$,
" " $CDO = y_1$,
" " $CDB = y_2$,
" " $ACO = z_1$,
" " $ACD = z_2$,
" " $AOB = r$,
" " $AOD = s$,
" " $AOC = t$,

and the reading of A from O when the cross wires of the theodolite were returned back $= v$.

Let also n = excess of the sum of the observed angles of the quadrilateral figure $ABDC$ over 360°,

n_1 = excess of the observed angles of the triangle AOB over 180°
n_2 = " " " BOD "
n_3 = " " " DOC "
n_4 = " " " COA "

from which we have

$$w_1+x_2-x_1+r=n_1 \text{ ....}(1)$$
$$x_1+y_2-y_1+s-r=n_2 \text{ ....}(2)$$
$$y_1+z_2-z_1+t-s=n_3 \text{ ....}(3)$$
$$z_1+w_2-w_1+v-t=n_4 \text{ ....}(4)$$

adding these four equations together,

$$x_2+y_2+z_2+w_2+v=n_1+n_2+n_3+n_4,$$

but
$$x_2+y_2+z_2+w_2=n \text{ ....}(5)$$

and consequently $v+n=n_1+n_2+n_3+n_4$. In order to find the values of the twelve quantities, w_1, w_2, $x_1 \dots r$, s, connected by the five equations (1), (2), (3), (4), (5), we must make seven probable assumptions. Let therefore

$$w_1=x_2-x_1;\ x_1=y_2-y_1;\ y_1=z_2-z_1;\ \text{and } z_1=w_2-w_1,$$

also $r=s-r$; and $s-r=t-s_1=v-t$, as the most probable assumptions we can make.

Hence from equation (1), $2w_1+\frac{v}{4}=n_1$,

$$\therefore\ w_1=\frac{n_1}{2}-\frac{v}{8}, \quad \therefore\ x_2-x_1=\frac{n_1}{2}-\frac{v}{8}$$

Similarly from (2), $x_1=\frac{n_2}{2}-\frac{v}{8}$, and $y_2-y_1=\frac{n_2}{2}-\frac{v}{8}$,

(3), $y_1=\frac{n_3}{2}-\frac{v}{8}$, and $z_2-z_1=\frac{n_3}{2}-\frac{v}{8}$,

(4), $z_1=\frac{n_4}{2}-\frac{v}{8}$, and $w_2-w_1=\frac{n_4}{2}-\frac{v}{8}$.

Hence

$$\left.\begin{array}{lll}
\text{correction to angles } BAO \text{ and } OBA \text{ are each} & & =\frac{v}{8}-\frac{n_1}{2} \\
\quad " \qquad DBO \text{ and } ODB & " & =\frac{v}{8}-\frac{n_2}{2} \\
\quad " \qquad CDO \text{ and } OCD & " & =\frac{v}{8}-\frac{n_3}{2} \\
\quad " \qquad ACO \text{ and } OAC & " & =\frac{v}{8}-\frac{n_4}{2}
\end{array}\right\} \text{.........}(c)$$

and correction to angles at $O=-\frac{v}{4}$ for each.

The following example will best illustrate this. Referring to figure—

		Angle	°	′	″
The angles observed at	A	$BAO=$	29°	40′	24″
		$OAC=$	30	39	37
" "	B	$DBO=$	57	37	45
		$OBA=$	74	19	16
" "	C	$ACO=$	73	2	10
		$OCD=$	41	32	33 ·7
" "	D	$CDO=$	23	38	11 ·7
		$ODB=$	29	30	7 ·6
Sum of the angles of $ABDC$, -			360°	0′	5″

	Angle	°	′	″
The angles observed at O	$AOB=$	76°	0′	21″
	$BOD=$	92	52	10 ·5
	$DOC=$	114	49	18 ·5
	$COA=$	76	18	12
Sum, - - - - -	$=$	360°	0′	2″

		°	′	″
Triangle AOB	angle $BAO=$	29°	40′	24″
	" $OBA=$	74	19	16
	" $AOB=$	76	0	21
Sum of the angles, - -		180°	0′	1″

		°	′	″
Triangle BOD	angle $DBO=$	57°	37′	45″
	" $ODB=$	29	30	7 ·6
	" $BOD=$	92	52	10 ·5
Sum of the angles, - -		180°	0′	3″·1

		°	′	″
Triangle COD	angle $CDO=$	23°	38′	11″·7
	" $OCD=$	41	32	33 ·7
	" $DOC=$	114	49	18 ·5
Sum of the angles, - -		180°	0′	3″·9

		°	′	″
Triangle AOC	angle $ACO=$	73°	2′	10″
	" $OAC=$	30	39	37
	" $COA=$	76	18	12
Sum of the angles, - -		179°	59′	59″

Here $n_1=1''$, $n_2=3''{\cdot}1$, $n_3=3''{\cdot}9$, $n_4=1''$, $v=2''$, and $n=7''$.

Substituting these values for the symbols in (C), we find

$$\text{Correction for angles} \left\{ \begin{matrix} BAO \\ \text{and} \\ OBA \end{matrix} \right\} = 0''{\cdot}25 - 0''{\cdot}5 \;\; = -0''{\cdot}25$$

$$\text{,,} \quad \text{,,} \quad \left\{ \begin{matrix} DBO \\ \text{and} \\ ODB \end{matrix} \right\} = 0''{\cdot}25 - 1''{\cdot}55 = -1''{\cdot}3,$$

$$\text{,,} \quad \text{,,} \quad \left\{ \begin{matrix} CDO \\ \text{and} \\ OCD \end{matrix} \right\} = 0''{\cdot}25 - 1''{\cdot}95 = -1''{\cdot}7$$

$$\text{,,} \quad \text{,,} \quad \left\{ \begin{matrix} ACO \\ \text{and} \\ OAC \end{matrix} \right\} = 0''{\cdot}25 + 0''{\cdot}5 \;\; = 0''{\cdot}75.$$

Applying these corrections we have

$$\triangle AOB \left\{ \begin{matrix} \angle A = & 29^\circ & 40' & 23''{\cdot}75 \\ \angle B = & 74 & 19 & 15\ {\cdot}75 \\ \angle O = & 76 & 0 & 20\ {\cdot}5 \end{matrix} \right. \qquad \triangle BOD \left\{ \begin{matrix} \angle B = & 57^\circ & 37' & 43''{\cdot}7 \\ \angle D = & 29 & 30 & 6\ {\cdot}3 \\ \angle O = & 92 & 52 & 10 \end{matrix} \right.$$

$$\text{Sum,} \; - \; 180^\circ \;\; 0' \;\; 0'' \qquad\qquad \text{Sum,} \; - \; 180^\circ \;\; 0' \;\; 0''$$

$$\triangle COD \left\{ \begin{matrix} C = & 41^\circ & 32' & 32'' \\ D = & 23 & 38 & 10 \\ O = & 114 & 49 & 18 \end{matrix} \right. \qquad \triangle AOC \left\{ \begin{matrix} A = & 30^\circ & 39' & 37''{\cdot}75 \\ C = & 73 & 2 & 10\ {\cdot}75 \\ O = & 76 & 18 & 11\ {\cdot}5 \end{matrix} \right.$$

$$\text{Sum,} \; - \; 180^\circ \;\; 0' \;\; 0'' \qquad\qquad \text{Sum,} \; - \; 180^\circ \;\; 0' \;\; 0''$$

$AB = 36527$ feet—

log AB - - - -	4·5626140	- - - - -	4·5626140
log sin (29° 40′ 23″·75)-	9·6946516	log sin (74° 19′ 15″·75) -	9·9835321
log cosec (76 0 22 ·5)-	0·0130852	- - - - -	0·0130852
log BO = 18635·92 - -	4·2703508	log AO = 36243·6 - -	4·5592313
log AO - - - -	4·5592313	- - - - -	4·5592313
log sin (30° 39′ 37″·75)-	9·7075274	log sin (76° 18′ 11″·5) -	9·9874706
log cosec (73 2 10 ·75)-	0·0193198	- - - - -	0·0193198
log CO(=19323·18) -	4·2860785	log AC(=36814·73) -	4·5660217
log BO(=18635·92)			

Sum = 37959·1

Diff. = 687·20

$\angle BOC = 152^\circ\ 18'\ 32''$

$180^\circ - BOC = 27^\circ\ 41'\ 28''$ .... log *Diff.* 2·8370832

$90^\circ - \frac{1}{2}BA = 13^\circ\ 50'\ 44''$ ....log tan 9·3917584

		12·2288416
Or $\frac{1}{2}(OBC+BCO)=13°\ 50'\ 44''$	log *Sum* =	4·5793160
$\frac{1}{2}(OBC-BCO)=15\ \ 20\ \ \cdot34$	..........log tan	7·6495256

$$\angle OBC = 14°\ \ 6'\ \ 4''\cdot34$$
$$\angle BCO = 13\ \ 35\ \ 23\ \ \cdot66$$

Again, using equation (3) on p. 196, we have

log *CO* - - - -	4·2860785	log *BO* - - - -	4·2703508
log cos (13° 35′ 23″·6) -	9·9876674	log cos (14° 6′ 4″·34) -	9·9867121
log 18782·18 - - -	4·2737459	log 18074·36 - - -	4·2570629
18074·36			
36856·54 = *BC*			

$$\therefore\ \angle ABC = 74°\ 19'\ 15''\cdot75 - (14°\ \ 6'\ \ 4''\cdot34) = \ 60°\ 13'\ 11''\cdot41$$
$$\angle BAC = 29\ \ 40\ \ 23\ \cdot75 + (30\ 39\ \ 37\ \cdot75) = \ 60\ \ 20\ \ \ 1\ \cdot5$$
$$\angle BCA = 73\ \ \ \ 2\ \ 10\ \cdot75 - (13\ 35\ \ 23\ \cdot66) = \ 59\ \ 26\ \ 47\ \cdot09$$

$$\text{Sum, - - - -} = 180\ \ \ 0\ \ \ 0$$

log *AB* - - - -	4·5626140	- - - - - -	4·5626140
log sin (60° 20′ 1″·5)	9·9389814	log sin (60° 13′ 11″·41) -	9·9384884
log cosec(59 26 47 ·09)	0·0649191	- - - - -	0·0649191
log *BC*(=36856·535) -	4·5665145	log *AC*(=36814·72) -	4·5660215

The agreement of these values of *BC* and *AC* serves to confirm their accuracy, and we may consider the position of the station satisfactorily established and proceed to determine *D* from *B* and *C*.

$$\angle DBC = 57°\ 37'\ 43''\cdot7\ \ + 14°\ \ 6'\ \ 4''\cdot34 = \ 71°\ 43'\ 48''\cdot04$$
$$\angle BCD = 13\ \ 35\ \ 23\ \cdot66 + 41\ \ 32\ \ 32\ \ \ \ = \ 55\ \ \ 7\ \ 55\ \cdot66$$
$$\angle CDB = 23\ \ 38\ \ 10\ \ \ \ + 29\ \ 30\ \ \ 6\ \cdot3\ = \ 53\ \ \ 8\ \ 16\ \cdot3$$

$$180°\ \ 0'\ \ 0''$$

log *BC* - - - -	4·5665145	- - - - - -	4·5665145
log sin (55° 7′ 55″·66) -	9·9140641	log sin (71° 43′ 48″·04) -	9·9775360
log cosec (53 8 16 ·3) -	0·0968659	- - - - -	0·0968659
log *BD*(=37795·89) -	4·5774445	log *CD*(=43743·79) -	4·6409164

log OB - - - -	4·2703508
log sin (92° 52′ 10″) -	9·9994551
log cosec (29 38 6 ·3) -	0·3076377
log BD(=37795·81) -	4·5774436

The agreement of these two values of BD serves to confirm the position of D.

If O cannot be seen from D, and *vice versa*, the angles AOD and CDO, and their errors, must be omitted from the former example; and in this case the quadrilateral figure $OBDC$ gives

$$x_1+y_2+z_2-z_1+t-r=n_2+n_3, \qquad \text{.................(7)}$$

together with equations (1), (4), (5), and (6), which still hold.

Assuming $r=t-r=v-t$, we have

$$r=\frac{v}{3};\ t-r=\frac{v}{3};\ \text{and}\ v-t=\frac{v}{3}.$$

Assuming, as before, $w_1=x_2-x_1$ and $z_1=w_2-w_1$, we find

$$w_1=x_2-x_1=\frac{n_1}{2}-\frac{v}{6},$$

$$z_1=w_2-w_1=\frac{n_4}{2}-\frac{v}{6}.$$

Assume $x_1=y_2=z_2-z_1$, when (7) gives

$$3x_1+\frac{v}{3}=n_2+n_3,$$

$$x_1=\frac{n_2+n_3}{3}-\frac{v}{9};$$

We may here observe that n_2+n_3 is merely the excess of the sum of the observed angles of the quadrilateral figure $OBDC$ above 360°, and may be expressed by a single symbol, instead of the sum of two; hence

Correction to angle $BAO=\frac{2}{9}''-\frac{1}{2}''=-0''\cdot17$,
" " $OBA=\frac{2}{9}-\frac{1}{2}=-0\cdot17$,
" " $DBO=\frac{2}{9}-\frac{7}{3}=-2\cdot1$,
" " $CDB=\frac{2}{9}-\frac{7}{3}=-2\cdot1$,
" " $OCD=\frac{2}{9}-\frac{7}{3}=-2\cdot1$,
" " $ACO=\frac{2}{9}+\frac{1}{2}=0\cdot83$,
" " $OAC=\frac{2}{9}+\frac{1}{2}=0\cdot83$.

Corrections for the angles observed at $0=-\frac{2}{3}''=-0''\cdot66$.

Hence the corrected angles are

$$\text{For triangle } AOB \begin{cases} \text{Angle } A = 29^\circ\ 40'\ 24'' - 0''\cdot17 = 29^\circ\ 40'\ 23''\cdot83 \\ \quad ,, \quad B = 74\ \ 19\ \ 16\ - 0\ \cdot17 = 74\ \ 19\ \ 15\ \cdot83 \\ \quad ,, \quad O = 76\ \ \ 0\ \ 21\ - 0\ \cdot66 = 76\ \ \ 0\ \ 20\ \cdot34 \end{cases}$$

$$\text{Sum, - - - } 180^\circ\ \ 0'\ \ 0$$

$$\text{For triangle } AOC \begin{cases} \text{Angle } A = 30^\circ\ 39'\ 37'' + 0''\cdot83 = 30^\circ\ 39'\ 37''\cdot83 \\ \quad ,, \quad C = 73\ \ \ 2\ \ 10\ + 0\ \cdot83 = 73\ \ \ 2\ \ 10\ \cdot83 \\ \quad ,, \quad O = 76\ \ 18\ \ 12\ - 0\ \cdot66 = 76\ \ 18\ \ 11\ \cdot34 \end{cases}$$

$$\text{Sum, - - - } 180^\circ\ \ 0'\ \ 0''$$

From these we find

$$BO = 18635\cdot937, \quad AB = 36243\cdot61,$$
$$CO = 19323\cdot19, \quad AC = 36814\cdot73.$$

$$\angle\ OBC = 14^\circ\ 6'\ 4''\cdot56 \qquad \text{Triangle } ABC \begin{cases} \text{Angle } A = 60^\circ\ 20'\ 1''\cdot66 \\ \qquad\ B = 60\ \ 13\ 11\ \cdot27 \\ \qquad\ C = 59\ \ 26\ 47\ \cdot07 \end{cases}$$
$$BCO = 13\ 35\ 23\ \cdot76$$

$$\text{Sum, - - - } = 180^\circ\ \ 0'\ \ 0''$$

$$BC = 36855\cdot543, \quad AC = 36814\cdot693,$$
$$BD = 37795\cdot76. \quad AD = 43743\cdot64.$$

Comparing this with the former, where D and O were in sight of each other, we see the difference is slight, and may feel satisfied that the foregoing method gives as accurate a determination as is possible under the circumstances.

We now come to secondary stations placed to fill in the larger details of the space occupied by a main triangle which therefore depend upon the three main stations for their projection. The straight lines from them should intersect at good angles; they should, if possible, be placed in sight of the three main stations, or at least of two of them, the lines from which should cut at right angles, or as nearly so as possible; they should be selected from advantageous positions for seeing the country immediately surrounding them. Tertiary stations placed for filling in the more minute detail should be selected in good places for seeing from and being seen from the parts of the country near them to enable the country to be sketched in correctly; the angle between a main and secondary station or two secondary stations in sight of a tertiary station, which are to be used to determine its position, should subtend at it an angle as nearly equal to 90° as possible.

The stations having been selected and made, the angles should be observed at the main stations with the greatest care and precision as regards the main stations depending on it, and those on which it depends; all such angles should be equally well observed. To do this the observer, having set up the best theodolite the survey can command, and made its adjustments with the greatest possible care, notes in his angle book the main stations in sight in the order they stand from left to right, the station he intends for the zero mark of the round being noted first; the verniers must then be carefully read and noted in the angle book, the cross of the telescope wires then made to intersect the zero station accurately, and the lower plate firmly clamped; this being satisfactorily accomplished, unclamp the upper plate, and moving the upper plate from left to right observe each main station in succession with the greatest possible care, and note the readings of the verniers, until the cross of the telescope wires comes round to the zero mark, which must be carefully observed, and the readings of the verniers noted. If these readings agree with, or differ but slightly from the former zero readings, and the levels have kept in the same state near their normal positions as at starting, or nearly so, the round of angles may be considered satisfactory, otherwise the observation must be repeated. A good theodolite properly handled generally returns to its zero after a round of angles have been observed sufficiently near to be satisfactory; an instrument which in good hands does not return well to its zero, and will not keep its adjustments, ought not to be used.

Having satisfactorily observed the first round of angles, the observer should take for a new zero the main station to the right, the reading of the A vernier corresponding to which in the angle book is nearest 120°. Commencing with it, observe a round of angles to the same main stations with equal care and precision, noting the readings of the verniers in the angle book. This round proving satisfactory, the observer should select from the stations the one which in the first round of angles read nearest to 240°, and taking it for a third zero mark, observe a third round in a similar manner to the same objects; the mean of the three results thus obtained will give the value of each angle with great accuracy.

The secondary stations proper to be taken from the observer's position must be selected and noted in order in the angle book, including with them in their proper places the primary stations already observed. The secondary stations must be selected as follows:—First, those which are nearer to the observers' station than they are to any other main station;

Secondly, those between the observer's station and any of the other main stations at which the angle made by the straight lines drawn from the two primary stations to the secondary station cut at right angles, or nearly so; this angle should not in any case be less than 30°, nor exceed 150°. Thirdly, those that are between the observer and a main station and nearly in line with it.

Two rounds of angles to these objects will be sufficient, the zero of the theodolite being set successively at two main stations, and the observer's station being between them and as nearly as possible on the line joining them. First one is taken for the zero, and each station, primary and secondary, observed in order, and the readings noted, and then the other main station is taken as the zero mark, and a similar round to the same objects carefully observed and noted.

When the north point of the magnetic needle reads on its own graduated circle exactly 90° or 270° as the case may be, the reading of the A vernier of the theodolite must be carefully taken and noted, the tangent screw of the upper plate being used to eliminate the friction of the pivot point about which the needle turns, in the manner already described.

Lastly, the tertiary objects near the station, and those at which the station subtends a good angle with a secondary station to be used to determine with it the position of the tertiary station or object, must be observed, as well as the secondary stations upon which they depend taken in order, the reading of the A vernier only being noted: in doing this the angle of elevation given by the vertical circle must be carefully read and noted for each object, as well as the reading of the telescope bubble in its central position; the horizontal wire must be made to touch the ground on which the station stands each time before reading the vertical circle, whilst the vertical wire passes through the centre of the station.

The angles to the main and secondary station, if any, which are intended to be used to determine the position of a secondary station, should be noted in the angle book by the observer before he commences his observations, as well as those secondary stations which will depend upon the station at which he is observing. The zero of the theodolite being set at the most distinct and distant station to be observed, he must take two rounds of angles to these points in the same manner as the secondary stations were observed from a primary station; and after these observations have been carefully made and noted, he must take a round of angles with the A vernier readings to the tertiary objects, and the primary and secondary

stations to be used for their projection; and the angles of elevation in the same way as those at the primary stations were taken.

It will only be necessary to observe a few angles from a tertiary station, viz., those for its own determination between the nearest primary and secondary stations upon which it depends, and the tertiary objects near it. When a theodolite is used, which it should be at the most important of this class of stations, the A vernier only need be read, but the readings of the vertical circle must be carefully taken and noted. Frequently a sextant will answer the purpose perfectly well, and, when using it, a distant well-defined object should be selected, to which the angles of the other object must be referred, taking them in order—the angle between the zero object thus selected and that nearest to it being taken first. When the angle becomes too large for the sextant, another distant zero mark must be selected, and the angle between it and the first zero mark observed and noted. Three such zero marks will carry the angles round the circle, and their sum ought to be 360°; if not, the angles must be corrected by applying to each of them one-third of the difference between 360° and the sum of the three angles; but when using the difference between two angles referred to the same zero mark it is obvious that this correction will disappear and must not be applied.

Suppose a secondary station on the coast line (as *C*, Fig. 32)

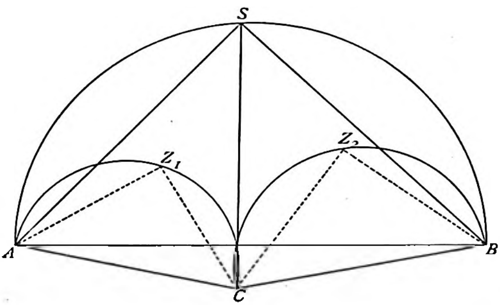


Fig. 32.

to be so situated that it can be seen only from two primary stations *A* and *B* on the coast line between which it lies, but the lines from *A*, *B* to it cut too obliquely to determine its position with sufficient accuracy, and it cannot be seen from any of the inshore stations, while *A*, *B*, and *C* not only see each

other but have a clear view out to sea. In this case the vessel should be laid off as nearly as possible to the point S on the straight line CS bisecting the angle ACB, where the angle ASB is $90°$; theodolites being placed at A, B, and C, and observations made in a similar manner to those already described for determining a primary station under similar circumstances.

Tertiary stations so situated between A and B must be cut off from the ship with a sextant at the same time; when this plan is not available a boat fixed independently by means of the three points A, B, and C should be used, being placed at Z_1, for cutting off the tertiary marks between A and C, where Z_1 is as nearly equidistant from A and C as can be estimated with the eye, and the angle $AZ_1C = 90°$, or as near thereto as possible.

To cut off the tertiary marks between C and B the boat should be placed as near as possible to Z_2, where $Z_2C = Z_2B$, and the angle $BZ_2C = 90°$ (see Fig. 32).

When projecting these observations it is important to know from which of the straight lines, drawn from the boat's position on the sheet to that of a known station on shore, the sextant angle to one of these tertiary marks should be laid off, so that an error in the boat's position may produce the smallest possible error in the projection of the point.

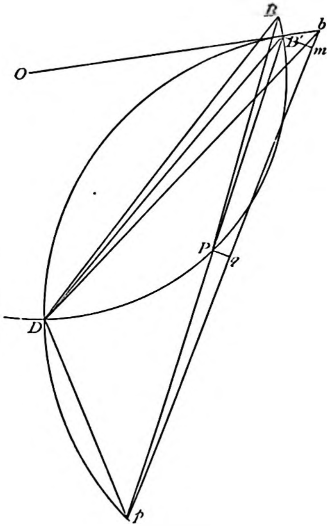


Fig. 33.

Let B (Fig. 33) be the boat, D a known point on the shore, and P a point on the sea-beach whose position is to be determined by means of the angle PBD observed with a sextant from the boat. Where ought D to be, in order that the error in P's projection arising from an error in the projection of B may be the smallest possible.

Let B be the correct position of the boat or vessel, b its erroneous position. Join bD, and make the angle Dbp equal to the observed angle B; from

P draw Pq perpendicular to bp; then the shorter Pq is the nearer the straight line bp will pass P, and the better will be the situation of D.

Let O be the centre of the circle described through the points B, P, and D, join Ob cutting the circumference when produced, if necessary, in B', through the three points b, B', and D, describe a circle of which $bB'Dp$ is the circumference, join DB, DB', PB, and PB'. The angle $DB'P$ = angle DBP in the same segment of the circle = observed angle $B = Dbp$; therefore the straight line $B'P$, if produced, will meet the straight line bp in a point p in the circumference of the circle $bB'Dp$: because the equal angles $DB'P$, Dbp, having a common point D in the circumference of the circle $bB'Dp$, in which their angular points B' and b respectively lie, must have a common point p in which their other sides containing them meet also in the circumference of the same circle.

From B' draw $B'm$ parallel to Pq, and therefore perpendicular to bp.

Because $pDB'b$ is a quadrilateral figure inscribed in a circle,

the angle $\qquad Dpb$ + angle $DB'b = 180°$;

also angle $\qquad OB'D$ + „ $DB'b = 180$;

$$\therefore \text{ angle } Dpb = \text{angle } OB'D.$$

Again, because O is the centre of the circle $DPB'B$,

$$\text{Angle } DPB' = \text{angle } OB'D + 90° = \text{angle } Dpb + 90°;$$

but $\qquad$ „ DPB' = „ PDp + angle DpP;

$$\therefore \text{ angle } PDp' + \text{angle } DpP = \text{angle } Dpb + 90°.$$

$$\therefore \text{ angle } PDp = 90° + \text{angle } Dpb - \text{angle } DpP = 90° - \text{angle } B'pb.$$

Since the angle $B'pb$ is necessarily very small, we may consider the angle $PDp = 90°$, without introducing a sensible error in the result we are aiming at; and upon this supposition we shall have

$$Pp = \frac{PD}{\sin DpP}.$$

Now, $$B'p = B'D\,\frac{\sin B'Dp}{\sin DpP} = B'D\,\frac{\sin B'bp}{\sin DpP},$$

since $$B'Dp + B'bp = 180°;$$

$$\therefore \frac{Pp}{B'p} = \frac{PD}{B'D \sin B'bp};$$

but $B'm$ being parallel to Pq,

$$\frac{Pq}{B'm} = \frac{Pp}{B'p} = \frac{PD}{B'D \sin B'bp};$$

$$\therefore\ Pq = \frac{PD}{B'D}\ \frac{B'm}{\sin B'bp} = \frac{PD}{B'D} \cdot B'b$$

$$= \frac{PD}{BD} \cdot B'b \text{ very approximately.}$$

$B'b$ depends upon the unknown error in projecting the boat, and is therefore unknown and uncontrollable; therefore we must take B so that $\frac{PD}{BD}$ is as small as possible, which will be the case when D is as close to P as possible, and as far from B

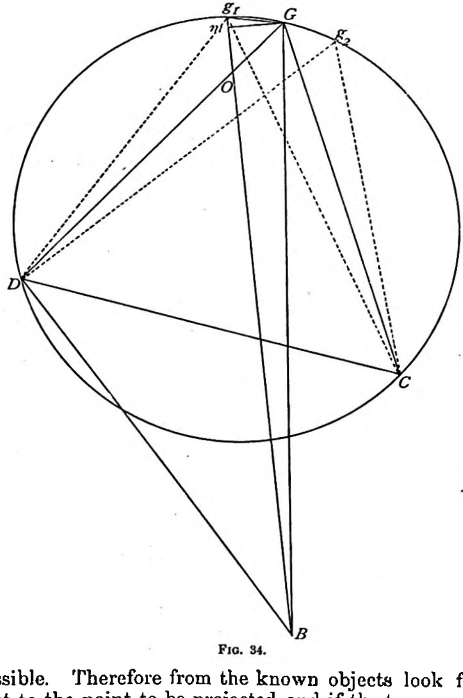


Fig. 34.

as possible. Therefore from the known objects look for those nearest to the point to be projected, and if the two nearest to P be equally distant from it, take that which is the farther from the boat.

When a conspicuous object is so situated as to be of primary importance, and the vertical axis of the theodolite cannot be made to pass exactly through the centre of its figure, the theodolite must be placed as near to the centre as possible, and the horizontal distance of the axis of the theodolite from the point of the object over which the theodolite would have been set up, if possible, measured with great care, and the reading of the A vernier of the upper plate read when the cross wires of the telescope intersect this spot when taking the first round of angles with the theodolite.

Let G (Fig. 34) be the centre of such an object which has been observed from two main stations C and D, whereby the angles DCG and GDC are known; there are two points, such as g_1 and g_2, one on each side of G, from which D and C can be seen, and where the angles Cg_1D and Cg_2D are each equal to CGD. A point of this description can easily be found with a pocket sextant near enough for all practical purposes, as follows: To find g_1, for instance, set the index of the sextant to the number of degrees and minutes in the angle GDC, which is known from the observations made at D; keep as near G as convenient for setting up the theodolite, so that it may have a good command over the surrounding objects. Starting from a position on the line DG, move from C slowly round G until the image of C reflected coincides with the centre of G seen directly. When this happens the observer will be at g_1, or sufficiently near for practical purposes. Join g_1D, g_1C, and g_1G; because the angle Gg_1C is equal to the angle GDC, the circumference of the circle passing through G, D, and C will also pass through g_1. Describe this circle, and we see that the angles Cg_1D, CGD in the same segment of the circle must be equal to each other.

In a similar manner a point g_2 on the opposite side of G, or between G and C, may be found; it is not absolutely necessary to place the theodolite exactly over g_1 or g_2; but the angle CGD being a very important one when near these points, the correction to the observed angle, to make it equal to that which would have been observed at G if the theodolite could have been placed there, becomes very small, and any error in the distance and direction of the axis of the theodolite from G will not sensibly affect the angle deduced from the observation so taken.

Let B be some other station; join DB, g_1B, and GB by straight lines, the two latter of which cut each other in the point O, and from G draw Gm perpendicular to g_1B.

Let also

$$g_1G = t,\ GD = b,\ GB = d,\ Gg_1B = \gamma_1,\ Bg_1D = \theta,$$

$$\text{angle } Bg_1D + \text{angle } g_1DG = \text{angle } g_1OG$$
$$= \text{angle } DGB + \text{ angle } g_1BG;$$

$$\therefore \quad \text{angle } DGB = \text{angle } Bg_1D + \text{angle } g_1DG - \text{angle } g_1BG$$
$$= \theta + g_1DG - g_1BG \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\sin g_1BG = \frac{Gm}{GB} = \frac{g_1G \sin Gg_1B}{GB} = \frac{t}{d}\sin\gamma_1 \ldots\ldots\ldots\ldots(2)$$

similarly $$\sin g_1DG = \frac{t}{b}\sin(\gamma_1+\theta) \ldots\ldots\ldots\ldots(3)$$

The values of g_1BG and g_1DG being determined from equations (2) and (3), and introduced into equation (1), will give the angle DGB.

If B be the zero point of the instrument, the angle g_1BG will be constant for all the readings of the instrument; the reading of G as drawn in the figure will be $360° - \gamma_1$, the reading D will be θ, and $\sin(\theta + \gamma_1) = \sin(360° - \theta - \gamma_1)$

= sin (difference of the reading of the centre of the station and the object).

$\therefore \quad \sin g_1DG = \frac{t}{b}$ sin (difference between the theodolite readings of G and D)$\ldots\ldots\ldots\ldots(4)$

Consequently, after calculating the value of GBg_1 once, we shall only have to calculate that of G_1DG from equation (3) corresponding to each succeeding value of θ.

Take the following example in which $g_1G = 12{\cdot}75$ feet very carefully measured.

Objects.	Approximate Distance.	Reading of the A Vernier to nearest minute.		Correction to Theodolite Reading.	
	Feet.	°	′	Angle. ′	″
Zero B, - - - - -	17250	360	0		
Episcopal church spire, - -	9720	40	16	−2	20·4
Main station C, - - -	16790	60	35	−0	13
Castle flagstaff, - - -	6750	98	2	−0	36
Centre of station G, - -	exact 12·75	122	59		
Presbyterian church spire, -	7230	160	29	+5	49·5
Station D, - - - -	20750	185	0	4	0
Station E, - - - -	17390	294	16	2	31
Zero—back, - - - -	...	360	0		

Equation (2) gives $\sin g_1BG = \frac{t}{d}\sin\gamma_1$.

Here $t-12{\cdot}75$ feet, $d=17250$ feet, and $\gamma_1=360°-122°\ 59'$
$=237{\cdot}1$

log 12·75 -	-	1·10551
log sin 57° 1′	-	9·92367
		11·02918
log 17250 -	-	4·23679
log sin (2′ 8″)	-	6·79239 $\therefore$ angle $g_1BG=2'\ 8''$.

Equation (4) gives $\sin g_1DG=\frac{t}{b}\sin$ (difference in readings of G and D).

$\therefore$ log 12·75 - 1·10551 $\therefore \sin g_1\text{Sp.}G=\frac{12{\cdot}75}{9720}\sin(122°\ 59'-40°16')$

logsin(82° 43′) 9·99648

$=\frac{12{\cdot}75}{9720}\sin(82°\ 43')$

11·10199

log 9720 - 3·98767 $\therefore$ $g_1\text{Sp.}G=-4'\ 28''{\cdot}4$

logsin(4′ 28″·4) 7·11432 $g_1BG=\ \ 2\ \ 8$

Correction to Bg_1 E. Spire, - $=-2'\ 20''{\cdot}4$

Therefore, if we apply this correction to the reading of the theodolite placed at g_1 with its zero at B, it will be the same reading that would have been obtained could the theodolite have been placed at G with its zero laid at B.

The other corrections in the table have been determined in the same manner.

CHAPTER VIII.

COMPARISON OF THE ASTRONOMICAL AND GEODETIC MEASURES.

THE Triangulation and Astronomical observations described in the previous chapters, having been completed, must be brought together and made consistent with each other, so as to give each observation its fair and proper effect on the whole. We have, therefore, to consider the effect the figure of the earth has upon calculations for latitude, longitude, and bearings made under

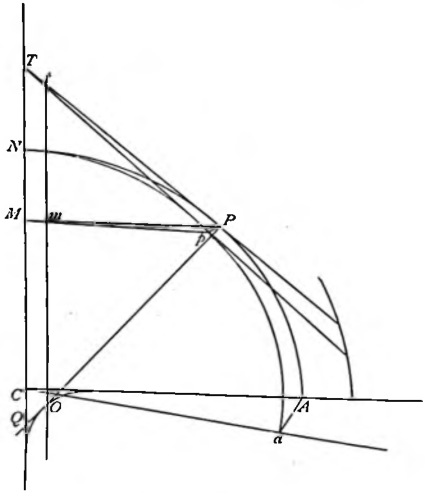


FIG. 35.

the supposition of its being a sphere. When careful measurements are made, the differences of longitude calculated on the spherical hypothesis are always larger than those given by

chronometers; this arises from the earth not being exactly spherical, but very nearly a spheroid generated by an ellipse of small eccentricity about its minor axis, which coincides with that of the earth's rotation; consequently calculations made on the spherical hypothesis require small corrections which we will now proceed to determine.

Let NPA (Fig. 35) be the meridian of P, a place on the surface of the earth; CN the semi-polar axis of the earth, and CA its equatorial semi-axis; O the centre of curvature of the elliptic meridian NPA at P; p a point on the earth's surface near P, and on the same parallel of latitude.

Let $CA=a$, $CN=b$, the eccentricity of the elliptic meridian $NPA=e$, the latitude of $P=l$; draw PM perpendicular to CN, and from O the centre of curvature of the meridian at P draw Omt parallel to CMN, cutting PM in m; join PO by a straight line, and produce it to cut NC produced in Q; join pM and pm; from P and p draw the tangents PT, pT to their respective meridians, cutting the axis CN produced in T, and let PT cut Omt in t; join pt.

Let D = difference of longitude between P and p.
D' = " " when calculated on the spherical hypothesis.

Then $D'-D=$ angle $pmP-$ angle pMP.

$$=\frac{Pp}{Pm}-\frac{Pp}{PM}=\frac{Pp}{PM}\cdot\frac{PM-Pm}{PM} \text{ very approximately}$$

$$=D'\frac{PQ-PO}{PQ}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$=D'\frac{e^2\cos^2 l}{1-e^2\sin^2 l}.$$

Assuming $e^2=\frac{1}{150}$,

$$D'-D=D'\frac{\cos^2 l}{150-\sin^2 l},$$

$$\therefore D=D'\left\{1-\frac{\cos^2 l}{150-\sin^2 l}\right\}$$

$$=D'\left\{1-\frac{\cos^2 l\,.\,\sec^2\phi}{150}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

making $\sin^2\phi=\dfrac{\sin^2 l}{150}$.

It is convenient to tabulate the values of $\frac{\cos^2 l \,.\, \sec^2 \phi}{150}$ for values of l within which the survey is situated. Suppose these latitudes to be 36° and 38° 30′; for latitude 36°, $\sin^2\phi = \frac{\sin^2 36°}{150}$.

log sin 36°, - -	9·769219	log cos 36°, - -	9·907958
log sin²36°, - -	19·538438	log cos²36°, - -	19·815916
log 150, - -	2·176091	log 150, - -	2·176091
log sin²ϕ, - -	17·362347		17·639825
log sin ϕ, - -	8·681173	log sec²ϕ, - -	0·001000
log sec ϕ, - -	0·000500	log multiplier D', -	3·640825

Therefore for latitude 36° log multiplier D', - $= \bar{3}\cdot640825$
Similarly „ 36 30′ „ - $= \bar{3}\cdot635293$
„ „ 37 30 „ - $= \bar{3}\cdot623917$
„ „ 38 30 „ - $= \bar{3}\cdot612123$

From these the following table is constructed.

Latitude.	Log multiplier of D' for correction.	Latitude.	Log multiplier of D' for correction.	Latitude.	Log multiplier of D' for correction.
36° 0′	$\bar{3}$·64082	37° 0′	$\bar{3}$·62966	38° 0′	$\bar{3}$·61807
10	·63899	10	·62775	10	·61610
20	·63715	20	·62584	20	·61412
30	·63529	30	·62392	30	·61212
40	·63343	40	·62198		
50	·63155	50	·62003		

A similar correction must be applied to the inclination of the meridians at two places calculated on the spherical hypothesis. The angle pTP is the inclination of the meridians at P and p to each other, and the angle ptP is that given by the spherical hypothesis. By a similar process of reasoning it may be shown that

$$\text{angle } pTP = \text{angle } ptP\left\{1 - \frac{\cos^2 l}{150 - \sin^2 l}\right\} \quad \ldots\ldots\ldots\ldots(3)$$

and that the above table may be used to calculate the correction necessary to reduce the inclination of the meridians calculated on the spherical hypothesis to their more correct values.

Take the following example:—*B* bears from *A*, N. 45° 17′ 25″ E. true, and is distant therefrom 16·3075 miles, the latitude of *A* being 37° 15′ N.; four places of figures in the logarithms will be quite sufficient for our present purpose.

log distance 16·3075,	1·2124	log spherical inclin.,	0·9472
log sin Mercat. bearing,	9·8522	log multiplier from table,	$\bar{3}$·6257
log tan mid. latitude,	9·8826	log correction,	$\bar{2}$·5729
log spher. inclin. 8·853,	0·9472	correction, - -	0′·037

∴ {inclination of meridians spherical,}	8′·853
correction, -	−0·037
Inclination of the meridians at *A* and *B* to each other, -	8′·816

From *A* true bearing of *B*, - -	N. 45° 17′ 25″ E.
Inclination of the meridians, -	+8 49
At *B* true bearing of *A*, - -	S. 45° 26′ 14″ W.
Mercatorial bearing, - - -	45° 21′ 49″·5

log distance *AB*, -	1·2123874	do.,	1·2123874
log sin Merc. bear.,	9·8522248	log cos Merc. bear.,	9·8467103
log sec mid. latitude,	0·0996376	Differ. latitude,	1·0590977
log spher. dif. long.,	1·1642498	log spher. dif. long.,	11′·4574 N.
log multiplier, -	$\bar{3}$·62570	Latitude *A*,	=37° 15′·0000 N.
Elliptical correction,	$\bar{2}$·78995	Latitude *B*,	=37° 26′·4574 N.
		Mid. latitude,	=37° 20′ 44″

∴ Spherical difference of longitude,	$=14'\cdot5965$
Elliptical correction, - - - -	$-\ 0'\cdot0616$
Difference of longitude, - -	$14'\cdot5349$

We see therefore that in the latitude of 37°, and at some distance north and south of it, the difference of longitude given by the spherical hypothesis will exceed that given by observations of the heavenly bodies, combined with chronometers carried between the places, by about one second of time for each degree of longitude, and this agrees very nearly with what we find to be the case.

A and *O* (Fig. 36) are two astronomical stations connected by a series of triangles, of which *AC*, *CE*, *EF*, *FK*, and *KO* are the sides which form the immediate lines of junction.

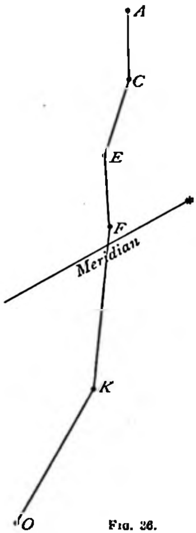


Fig. 36.

The measured base and triangulation gave

$AC=39613$	scale feet.	
$CE=46230\cdot4$	"	
$EF=41412\cdot3$	"	
$FK=92698$	"	
$KO=85070$	"	

angle $ECA=160°\ 20'\ 0''$
" $CEF=158\ \ 46\ \ 10$
" $KFE=170\ \ 9\ \ 5$
" $OKF=160\ \ 10\ \ 5$

By observations at *A*—True bearing of *CS* 57° 46′ 15″ E., probable size of error ±12″; Latitude by N. and S. stars, 36° 42′ 22″ N.; probable size of error ±1″·2.

By observations at *O*—True bearing of *K*, N. 29° 14′ 6″ W.; probable size of error ±14″·6; Latitude by N. and S. stars, 36° 7′ 13″ N.; probable size of error, ±1″·4. The meridian distance of *O* east of *A* was $2^m\ 50^s\cdot3$; the probable size of its error being $\pm0^s\cdot17$.

The probable size of the error is found by dividing the observations upon which the element depends into two sets or groups, each containing the same number of equally good observations taken in order without selection; the difference between the two values of the element given by the two groups divided by two is taken as the probable size of the error of the result given by the mean of all the observations.

Assuming that a mile of latitude in the middle latitude of AO contains 6080 scale feet—

Number of scale feet in $OK = 85070$, log =		4·9297764
" " one mile = 6080,	- -	3·7839036
log miles in OK (= 13′·991774),	- -	1·1458728

With this distance and the true bearing of K from O, the inclination of the meridians at K and O to each other is calculated in the same manner as previously described, and will be found to be equal to 4′ 59″·3. Hence we have

True bearing of K from O, - - -	N. 29° 14′ 6″ W.
Inclination of the meridians at K and O, -	4 59 ·3
At K, true bearing of O, - - - -	S. 29° 19′ 5″·3 E.
Angle OKF, - - - - - -	160 0 5
At K, true bearing of F, - - -	S. 130° 50′ 59″·7 W.

Calculating the inclination of the meridians at F and K to each other, we find it to be 8′ 28″·6. Applying this to the above we find that the true bearing of K from F is S. 49° 17′ 28″·9 E.

Calculating the inclination between the other meridians in the same way we find :—Inclination of the meridians at F and E to each other to be 4′ 18″·7; that between the meridians at E and C to be 3′ 27″·7; and that between the meridians at C and A, 4′ 5″·2; hence we have

True bearing of K from F, - - - -	S. 49° 17′ 28″·9 E.
Angle KFE, - - - - - -	170 9 5
True bearing of E from F, - - - -	S. 120° 51′ 36″·1 W.
	or N. 59 8 23 ·9 W.
Inclination of meridians at E and F, -	4 18 ·7
At E, true bearing of F, - - - -	S. 59° 12′ 42″·6 E.
Angle CEF, - - - - - -	158 46 10
At E, true bearing of C, - - - -	N. 37° 58′ 52″·6 W.
Inclination of meridians at C and E, -	3 27 ·7

At C, true bearing of E, - - -	S. 38° 2′ 20″·3 E.
Angle ECA, - - - - -	160 20 0
At C, true bearing of A, - - -	S. 122° 17′ 39″·7 W.
	or N. 57 42 20 ·3 W.
Inclination of meridians at A and C, -	4 5 ·2
At A, true bearing of C, - - -	S. 57° 46′ 25″·5 E.
„ „ by observations at A,	S. 57 46 15 E.
Difference, - - -	10″·5

Dividing this difference in proportion to the probable sizes of the errors of the observations for true bearing at the observing stations respectively, viz., as 12 : 14·6, we find that 4″·75 must be subtracted from the true bearing observed at A and all those depending on it, and 5″·75 must be added to the true bearing observed at O, and all the bearings depending on it must be corrected by the same amount.

Consequently the Mercatorial bearing of

	AC	becomes	S. 57° 44′ 17″·1 E.
„ „	CE	„	S. 38 0 30 ·6 E.
„ „	EF	„	S. 59 10 27 ·4 E.
„ „	FK	„	S. 49 13 8 ·8 E.
„ „	KO	„	S. 29 16 29 ·8 E.

From these and the distances expressed in miles we find as follows.

	East.		South.
AC gives departure,	5′·50944	Difference of latitude,	3′·47780
CE „	4 ·68218	„ „	5 ·99109
EF „	5 ·84901	„ „	3 ·49026
FK „	11 ·54476	„ „	9 ·95845
KO „	6 ·84199	„ „	12 ·20478
AO has for its depart.,	34′·42738	„ „	35′ 12238
log 35·12238, -	1·5455839	- - - -	1·5455839
log 34·42738, -	1·5369040	log cos Merc. bear.,	9·8537816
log cot Merc. bear.,	10·0086799	log AO (miles), -	1·6918023

Mercatorial bearing AO = S. 44° 25′ 38″·9 E. AO = 49′·18186.

By the astronomical observations

Latitude A, - -	36° 42′ 22″ N.	
„ O, - -	36 7 13 N.	
Difference of latitude,	35′ 9″	= 35′·15

Chronometers combined with astronomical observations at A and O gave O east of A 2^m $50^s·3$ or 42′·575.

The elliptical correction to be added = 0′·1843.

∴ Spherical difference of longitude = 42′·7593.

log 42·7593, -	1·6310305	log 35·15, - -	1·5459253
log cos mid. latitude,	9·9056648	log cos Merc. bear.,	9·8540590
log departure, -	1·5366953	log cos AO (miles),	1·6918663
log 35·15, - -	1·5459253		
		AO	= 49′·18881
log tan Merc. bear.,	9·9907700	Mer. br. AO = S. 44° 25′ 28″·3 E.	

Hence

Mercatorial bearing AO derived from observed true bearing and triangulations,	S. 44° 25′ 38″·9 E.
From the observed differences of latitude and longitude, - - - - - -	S. 44 23 28 ·3 E.
Difference, - - - - - - -	2′ 10″·6

These two values of the Mercatorial bearing of AO are derived from two sources perfectly independent of each other, and we may conclude that the accurate value of the bearing is somewhere intermediate between them, and differs from each by quantities proportional to the sizes of their probable errors.

The probable size of the errors in the bearings derived from the observations at A is ± 12″; and that of those derived from the observations at O ± 14″·6, and therefore the probable size of the error derived from their mean will be $= \pm\frac{12 \times 14·6}{26·6}$ = ± 6″·6.

To estimate the probable size of the error in the Mercatorial bearing of AO calculated from the astronomical differences of latitude and longitude, we see that the probable size of the error of the observed latitude of A is ± 1″·2; and that of the

observed latitude of O is $\pm 1''\cdot 4$; the error in the difference of latitude given by these observations will be the numerical sum or difference of their errors, according as they have the same or different signs, each of which is equally likely to happen; and consequently the most probable size of the error will be half the numerical sum of the two errors added to half their numerical difference; this will evidently be equal in size to the error which is numerically the larger of the two; therefore we take $\pm 1''\cdot 4$ as the probable size of the error in the difference of the observed latitudes of A and O, this expressed in miles of latitude is $\pm 0'\cdot 023$, the distance AO in miles to two places of decimals is $49'\cdot 18$; and therefore the probable size of the error in the Mercatorial bearing due to the error in the difference of latitude is, when expressed in circular measure, equal to $\pm \frac{0\cdot 023 \times 34\cdot 43}{(49\cdot 18)^2}$; the departure between the two places expressed in miles being $34'\cdot 43$. The probable size of the error in the astronomical difference of longitude between A and O is $\pm 0^s\cdot 17$; and therefore the probable size of the error in the Mercatorial bearing due to this error is, when expressed in circular measure, $= \pm \frac{0\cdot 0344 \times 35\cdot 12}{(49\cdot 18)^2}$, $35'\cdot 12$ being the difference of latitude between the two places. The combined effect of these two errors which are perfectly independent of each other will give an error the probable size of which is the same as the larger of the two numerically speaking; hence the size of the probable error in the Mercatorial bearing given by the astronomical differences of latitude and longitude will be $= \pm \frac{0\cdot 0344 \times 35\cdot 12}{(49\cdot 18)^2}$ expressed in circular measure. To calculate its value when expressed in minutes and seconds of angle—

log 0·0344, - - - - - -	$\bar{2}\cdot 5366$
log 35·12, - - - - - -	1·5456
	0·0822
2 log 49·18, - - - - - -	3·3836
log sin (1′ 43″) - - - - -	$\bar{6}\cdot 6986$

Hence the probable size of the error in minutes is $\pm 1'\ 43''$, that of the same bearing derived from the observed true bearings combined with the triangulation is $+6''\cdot 6$. We must therefore divide $3'\ 10''\cdot 6$ in the proportion of $103 : 6\cdot 6$ or as $2'\ 2''\cdot 7 : 7''\cdot 9$.

Hence Mercatorial bearing AO given by the astronomical differences of latitude and longitude, - - - -	S. 44° 23′ 28″ E
Correction, - - -	+2 2·7
Correct Mercatorial bearing, - - -	S. 44° 25′ 31″ E.
Mercatorial bearing of AO derived from observed bearings and triangulation,-	S. 44 25 38·9E.
Correction, - - -	−7·9
Correct Mercatorial bearing, - - -	S. 44° 25′ 31″ E.

Taking this and the distance $AO = 49'{\cdot}18881$ as correct, we calculate the differences of latitude and longitude thus:

log 49·18881, -	- 1·6918663	- -	- 1·6918663
log cos (44° 25′ 31″),	- 9·8537979	log sin, -	- 9·8450847
		log sec mid. lat.,	0·0943352
log dif. lat. (=35·12887),	1·5456642	log sph. dif. long.,	1·6312862
log spherical dif. long.,	1·63128	Sph. dif. long., =	42·78448
log elliptic. multiplier,	$\bar{3}$·63130	Ellip. cor.,	− 0·1852
		Differ. long.,	42·5993
log elliptic. correction, (=0′·1852),	- $\bar{1}$·26258	Differ. in time,	$2^m\ 50^s{\cdot}397$

Let y_1 be the error in the observed latitude of A necessary to make it consistent with the bearing and distance of O from A, which has just been determined, and y_2 the error in the observed latitude of O.

$$\text{The latitude of } A = 36°\ 42'{\cdot}36667 + y_1 \text{ N.}$$
$$\text{,,} \qquad O = 36 \quad 7{\cdot}21667 + y_2 \text{ N.}$$

$$\text{Difference of latitude} = 35{\cdot}15 + y_1 - y_2,$$
$$= 35{\cdot}12887,$$
$$\therefore y_2 - y_1 = 0{\cdot}02113.$$

Assuming that the errors bear the same proportion to each other as the probable sizes of the errors of observation in the latitudes of A and O respectively, but with opposite signs, we have, considering only their numerical values, $y_1 : y_2 :: 6 : 7$.

Hence $\qquad 7y_1 + 6y_2 = 0,$

but $\qquad 7y_2 - 7y_1 = 0'{\cdot}14791,$

$\therefore \qquad 13y_2 = 0'{\cdot}14791,$

$$\therefore \quad y_2 = \frac{0'\cdot 14791}{13} = 0'\cdot 01138,$$
$$y_1 = -0'\cdot 00975.$$

Therefore latitude $A = 36° \; 42'\cdot 36667 - 0'\cdot 00975$ N.
$= 36 \; 42\cdot 35692 = 36° \; 42' \; 21''\cdot 415$ N.
" $O = 36 \; 7\cdot 21667 + 0'\cdot 01138$ N.
$= 36 \; 7\cdot 22805 = 36° \; 7' \; 13''\cdot 683$ N.

The number of scale feet in a mile of latitude in the middle latitude of AO will therefore be equal to $6080 \frac{49\cdot 18156}{49\cdot 18881} = 6079\cdot 1$, and thus the most probable and consistent results of the observations made at and between the astronomical stations A and O are determined.

Suppose there be $\overline{n+1}$ consecutive astronomical stations, making n pairs of stations following each other, of which the Mercatorial bearings and distances have been determined in the manner just described. The direction of the meridian at the second astronomical station derived from the observations made at and between the first and second stations, when compared with that given by the observations made at and between the second and third astronomical stations, will not generally agree exactly with each other, but be inclined to each other by a small angle β_1 suppose, which we will consider positive when measured in the direction from north towards east, and so on. Also suppose $\pm p_1$ to be the probable size of the error in the direction of the meridian given by the first pair of astronomical stations, and $\pm p_2$ in that determined from the second pair.

Then the most probable direction of the meridian to be derived from the observations made at and between the three astronomical stations will be given by applying the correction $-\beta_1 \frac{p_1}{p_1+p_2}$ to the direction given by the first pair, and $\beta_1 \frac{p_2}{p_1+p_2}$ to that given by the second pair; or putting π_1 for $\frac{p_1}{p_1+p_2}$, the correction to the direction of the meridian given by the first pair of stations will be $-\beta_1\pi_1$, and that to the direction given by the second pair will be $\beta_1(1-\pi_1)$. Applying this correction, and comparing the direction of the meridian at the third astronomical station thus corrected with that derived from the observations made at and between the third and fourth astronomical stations, the size of the probable error of which is $\pm p_3$, suppose β_2 be the small angular difference

between the two directions; the correction to be applied to the direction of the corrected meridian will be $-\beta_2\frac{p_1p_2}{p_1p_2+p_1p_3+p_2p_3}$ or $-\beta_2\pi_2$, where $\pi_2=\frac{p_1p_2}{p_1p_2+p_1p_3+p_2p_3}$; and the correction to be applied to the direction of the meridian given by the third pair of astronomical stations will be $\beta_2(1-\pi_2)$.

Consequently the correction to direction
given by first pair of stations, - - $= -(\beta_1\pi_1+\beta_2\pi_2)$,
and to that given by second pair, - - $=\beta_1-(\beta_1\pi_1+\beta_2\pi_2)$.

Proceeding in this manner we find the correction to the direction of the meridian given by the fourth pair of astronomical stations will be $= \beta_3(1-\pi_3)$, where $\pi_3=\frac{p_1p_2p_3}{p_1p_2p_3+p_1p_2p_4+\dots+p_2p_3p_4}$, and the direction of the corrected meridian as given by the first three pairs of stations will require a further correction of $-\beta_3\pi_3$, making the correction to the direction given by the first pair of astronomical stations, - - - - - $= -(\beta_1\pi_1+\beta_2\pi_2+\beta_3\pi_3)$,
to that given by the second pair, - $=\beta_1-(\beta_1\pi_1+\beta_2\pi_2+\beta_3\pi_3)$,
" third pair, - $=\beta_2-(\beta_2\pi_2+\beta_3\pi_3)$,
and so on until we arrive at the nth pair, when the whole correction to the direction of the meridian given by first pair, - - - - $= -(\beta_1\pi_1+\beta_2\pi_2+\dots+\beta_{n-1}\pi_{n-1})$,
given by second pair, - $=\beta_1-(\beta_1\pi_1+\beta_2\pi_2+\dots+\beta_{n-1}\pi_{n-1})$,
" third pair, - $=\beta_2-(\beta_2\pi_2+\dots+\beta_{n-1}\pi_{n-1})$,
" nth pair, - $=\beta_{n-1}(1-\pi_{n-1})$,

$$\text{where } \pi_{n-1}=\frac{p_1p_2\cdots p_{n-1}}{p_1p_2\cdots p_{n-1}+p_1p_3\cdots p_{n-1}+\dots+p_2p_3\cdots p_n}.$$

To treat the distances between the astronomical stations in a similar manner, compare the distance a given station is from the second astronomical station, as determined from the observations made at and between the first and second astronomical stations, with the same distance determined from the observations made at and between the second and third astronomical stations, as follows. Suppose m_1 be the number of miles in the distance calculated from the observations corresponding to the first pair of stations, and m_2 that from the second pair of stations, and let $\mu_1=2\frac{m_1-m_2}{m_1+m_2}$; $\pm q_1$ the probable size of the error per mile made in first distance, and $\pm q_2$ that in the second; then the correction per mile to be applied to the distances given by the first pair of stations will

be $= -\dfrac{q_1\mu_1}{q_1+q_2} = -\delta_1\mu_1$ if we make $\delta_1 = \dfrac{q_1}{q_1+q_2}$; and the correction per mile to be applied to the distances given by the second pair of stations will $= \mu_1(1-\delta_1)$.

Proceeding in the same manner with the other pairs of stations we shall find the correction per mile to the distances given by the third pair of stations to be $\mu_2(1-\delta_2)$; where $\delta_2 = \dfrac{q_1q_2}{q_1q_2+q_1q_3+q_2q_3}$, and so on until we arrive at the nth pair, the correction to the distances given by the observations at and between which per mile will be expressed by $\mu_{n-1}(1-\delta_{n-1})$ where

$$\delta_{n-1} = \frac{q_1q_2 \cdots q_{n-1}}{q_1q_2 \cdots q_{n-1} + q_1q_3 \cdots q_n + \cdots + q_2q_3 \cdots q_n},$$

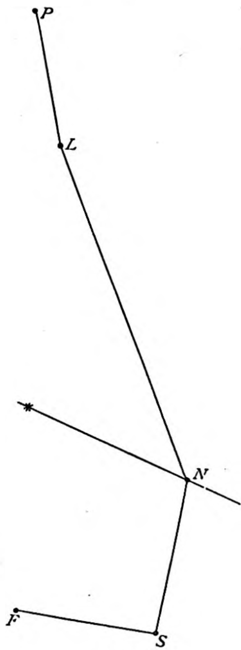


FIG. 37.

and the whole correction per mile to the distances given by the first pair of stations will

$= -(\mu_1\delta_1 + \mu_2\delta_2 + \ldots + \mu_{n-1}\delta_{n-1})$,

the correction for second pair

$= \mu_1 - (\mu_1\delta_1 + \mu_2\delta_2 + \ldots + \mu_{n-1}\delta_{n-1})$,

the correction for third pair

$= \mu_2 - (\mu_2\delta_2 + \mu_3\delta_3 + \ldots + \mu_{n-1}\delta_{n-1})$,

etc., etc.

the correction for nth pair

$= \mu_{n-1}(1-\mu_{n-1})$.

The following example shows how the foregoing is practically applied. P, L, N, S, and F (see Fig. 37) are five consecutive astronomical stations, the observations at and between which when calculated in the manner already explained gave the following results.

Latitude of $P = 44°\ 28'{\cdot}0357$ N. Mercatorial bearing of L = S. 56° 43′ 32″·2 W., the probable size of its error being ± 5″·8. Distance PL = 26′·0905, the probable size of its error per mile being ± 0′·000825. Latitude of L = 44° 13′·7212 N. Mercatorial bearing of N = S. 47° 10′ 13″·1 W., the probable size of its error = ± 6″·7. Distance LN = 63′·67607, the probable size of error per mile = ±0′·000325. Latitude of N = 43° 30′·4326 N.

Mercatorial bearing of $S=$S. 76° 47′ 51″·9 W., with probable size of its error $\pm 7''\cdot 8$. Distance $NS=30'\cdot 0636$, with size of its probable error per mile $= \pm 0''\cdot 000715$. Latitude of $S=43°$ 23′·5663 N. Mercatorial bearing of $F=$N. 14° 11′ 12″·1 W., probable size of error $\pm 4''\cdot 9$. Distance $SF=24'\cdot 65705$, probable size of error per mile $\pm 0'\cdot 00083$.

At L the distance of a station calculated from the observations made at and between P and L, was compared with the distance between same points, as calculated from the observations made at and between L and N, which gave $\mu_1=0\cdot 000154$; at N the comparison gave $\mu_2=-0\cdot 000087$; and at S it was found that $\mu_3=\cdot 000105$.

Comparing the Mercatorial bearings in the manner pointed out, and using the same notation, it was found that $\beta_1=-6''\cdot 3$, $\beta_2=+4''\cdot 75$, $\beta_3=-3''\cdot 9$.

Hence

$$\pi_1=\frac{5\cdot 8}{12\cdot 5}=0\cdot 464;\quad \pi_2=\frac{5\cdot 8\times 6\cdot 7}{5\cdot 8\times 6\cdot 7+5\cdot 8\times 7\cdot 8+6\cdot 7\times 7\cdot 8}=0\cdot 285,$$

and

$$\pi_3=\frac{5\cdot 8\times 6\cdot 7\times 7\cdot 8}{5\cdot 8\times 6\cdot 7\times 7\cdot 8+5\cdot 7\times 7\cdot 8\times 4\cdot 9+\ldots+6\cdot 7\times 7\cdot 8\times 4\cdot 9}=0\cdot 3129.$$

Therefore the correction to the Mercatorial bearing of

$$\begin{aligned}
PL&=6''\cdot 3\times 0\cdot 464-4''\cdot 75\times 0\cdot 285+3''\cdot 9\times 0\cdot 3129\\
&=2''\cdot 92-1''\cdot 35+1''\cdot 22=2''\cdot 8,\\
LN&=-6''\cdot 3+2''\cdot 8=-3''\cdot 5,\\
NS&=4''\cdot 75-1''\cdot 35+1''\cdot 22=4''\cdot 6,\\
SF&=-3''\cdot 9+1''\cdot 22=-2''\cdot 7.
\end{aligned}$$

Applying these corrections we find

Corrected Mercatorial bearing of	$PL=$	S.	56°	43′	35″	W.
„ „	$LN=$	S.	47	10	9 ·6	W.
„ „	$NS=$	S.	76	47	56 ·5	W.
„ „	$SF=$	N.	14	11	14 ·8	W.

and the same corrections must be applied to all the bearings found from the observations at and between each pair of astronomical stations respectively, and by so doing, the bearings throughout the whole series will be made consistent with each other, and have the most accurate values that can be obtained from the observations, each of which will have its full and proper influence on the whole.

We will now proceed to correct the distances; for this we have, referring to the previous notation,

$$q_1 = \pm 0'{\cdot}000825;\ \mu_1 = -0{\cdot}000154,$$
$$q_2 = \pm 0\ {\cdot}000325;\ \mu_2 = \quad 0{\cdot}000087,$$
$$q_3 = \pm 0\ {\cdot}000715;\ \mu_3 = -0{\cdot}000105,$$
$$q_4 = \pm 0\ {\cdot}000803,$$

$$\therefore \delta_1 = \frac{q_1}{q_1+q_2} = 0{\cdot}7173,$$

$$\delta_2 = \frac{825\times 325}{825\times 325+825\times 715+325\times 715} = 0{\cdot}2459,$$

$$\delta_3 = \frac{825\times 325\times 715}{825\times 325\times 715+825\times 325\times 803+825\times 715\times 803+325\times 715\times 803} = 0{\cdot}1796.$$

$\therefore$ correction to PL (distance per mile)

$$= \{0{\cdot}7173\times 0'{\cdot}000154$$
$$-0{\cdot}2459\times 0'{\cdot}000087+0{\cdot}1796\times 0{\cdot}000105\},$$
$$= 0'{\cdot}0001105-0'{\cdot}0000214+0'{\cdot}0000189,$$
$$= 0'{\cdot}000113,$$

„ $LN = -\{0'{\cdot}000154-0{\cdot}000113\}$
$= -0'{\cdot}000041,$

„ $NS = 0'{\cdot}000087-0'{\cdot}0000214+0'{\cdot}0000189,$
$= 0'{\cdot}0000845,$

„ $ST = -0'{\cdot}000105+0'{\cdot}0000189,$
$= -0'{\cdot}000086.$

log 26·0905 to four places of figures,	- -	1·4165
log 0·0001105 „	- -	$\bar{4}$·0531
log correction to $PL(=0'{\cdot}00295)$,	- -	$\bar{3}$·4696

$\therefore PL$	- - - -	$= 26'{\cdot}0905$
Correction,	- -	$+\ 0{\cdot}00295$
PL corrected	- -	$= 26'{\cdot}09345$

log 63·67607,	- - - -	1·8040
log 0·000041,	- - - -	$\bar{5}$·6128
log correction to $LN\ (=0'{\cdot}00261)$,	-	$\bar{3}$·4168

$\therefore LN$	- - - -	$= 63'{\cdot}67607$
Correction,	- -	$-\ 0{\cdot}00261$
LN corrected	- -	$63'{\cdot}67346$

log 30·0636, - - - -	1·4780
log 0·0000845, - - - -	$\bar{5}$·9269
log correction to *NS* (=0′·00254), -	$\bar{3}$·4049

∴ *NS* - - - -	=30′·0636
Correction, - -	+ 0 ·00254
NS corrected - -	=30′·06614

log 24·65705, - - - -	1·3919
log 0·000086, - - - -	$\bar{5}$·9345
log correction to *SF* (=0′·00212), -	$\bar{3}$·3264

∴ *SF* - - - -	=24·65705
- - - -	− 0·00212
SF corrected - -	=24·65493

With the corrected Mercatorial bearings and distances, the differences of latitude and longitude between the astronomical stations must be calculated, and their latitudes corrected so as to be consistent with each other.

For *PL* we have corrected Mercatorial bearing S. 56° 43′ 35″ W.
„ „ distance, - - 26′·09345.

log 26·09345, - -	1·4165315	- - - - -	1·4165315
log cos (56° 43′ 35″), -	9·7392857	log sin (56° 43′ 35″), -	9·9222375
		log sec (44 20 52 5), -	0·1456271
log diff. lat.(=14′·31585),	1·1558172		
Spherical differ. long., -	=30′·50676	log spherical differ. long.,	1·4843961
Elliptical correction, -	− 0 ·10433	log elliptical multiplier,	$\bar{3}$·5340101
Difference of long. *PL*,	30′·40243	log ellip. cor.(=0′·10433),	$\bar{1}$·018406

Corrected Mercatorial bearing of *LN*, - S. 47° 10′ 9″·6 W.
„ distance „ - 63′·67346

log 63·67346, - -	1·8039584	- - - - -	1·8039584
log cos (47° 10′ 9″·6), -	9·8324028	log sin (47° 10′ 9″·6), -	9·8653208
		log sec (43° 52′ 5″), -	0·1421023
log diff. lat.(=43′·28737),	1·6363612		
∴ Spher. differ. long.,	= 64′·77113	log spher. differ. long.	=1·8113815
Elliptical correction,	= − 0 ·22567	log elliptical multiplier	=$\bar{3}$·542094
Differ. of longitude *LN*,	64′·54546	log elliptical correction,	$\bar{1}$·353475

Corrected Mercatorial bearing of NS, S. 76° 47′ 56″·5 W.
„ distance „ 30′·06641.

log 30·06614, - -	1·4780777	- - - - -	1·4780777
log cos (76° 47′ 56″·5), -	9·3586391	log sin (76° 47′ 56″·5), -	9·9883694
		log sec mid. latitude, -	0·1390785
log diff. lat.(=6′·86621),	0·8367168		
Spher. differ. long. NS, =	40′·32048	log spher. differ. long., -	1·6055256
Elliptical correction, =	- 0·14212	log elliptical multiplier,	$\bar{3}$·547117
Differ. longitude NS, =	40′·17836	log elliptical correction,	$\bar{1}$·152642

Corrected Mercatorial bearing of SF, N. 14° 11′ 14″·8 W.
„ distance „ 24′·65493.

log 24·65493, - -	1·3918085	- - - - -	1·3919085
log cos (14° 11′ 14″·8), -	9·9865474	log sin (14° 11′ 14″·8), -	9·3893343
		log sec mid. latitude, -	0·1401002
log dif.lat. SF(=23′·90319),	1·3785559		
Spher. differ. longitude,	=8′·343398	log spher. differ. long., -	0·9213430
Elliptical correction,	- 0·029270	log elliptical multiplier,	$\bar{3}$·545079
Differ. of longitude SF,	=8′·314128	log elliptical correction,	$\bar{2}$·466422

To correct the observed latitudes—let y_1 be the error in the latitude of P, y_2 that of the latitude of L, y_3 in that of N's latitude, y_4 of S's, and y_5 of F's, and as the sizes of the probable errors of the observed latitudes were very nearly the same, we assume as probable that

$$y_1+y_2+y_3+y_4+y_5=0 \quad\text{..................... (1)}$$

Then latitude $P=44°\ 28'·0357+y_1$ N.
$L=44\ \ 13·7212+y_2$ N.
$N=43\ \ 30·4326+y_3$ N.
$S=43\ \ 23·5663+y_4$ N.
$N=43\ \ 47·4714+y_5$ N.

∴ Difference of latitude $PL=14'·3145+y_1-y_2$,
$=14·31585$;

$$\therefore\ y_1-y_2=0·00135 \quad\text{........................... (2)}$$

Difference of latitude $LN=43·2886+y_2-y_3$,
$=43·28737$;

$$\therefore\ y_3-y_2=0·00123 \quad\text{........................... (3)}$$

Difference of latitude $NS=6·8663+y_3-y_4$,
$=6·86621$;

$$\therefore\ y_4-y_3=0·00009 \quad\text{........................... (4)}$$

Difference of latitude $SF=23·9051+y_4-y_5$,
$=23·90319$;

$$\therefore\ y_4 - y_5 = 0{\cdot}00191 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$\therefore$ Subtracting (3) from (2), $y_1 - y_3 = 0'{\cdot}00012$.
„ (4) from above, $y_1 - y_4 = 0{\cdot}00003$.
Adding (5) to above, $y_1 - y_5 = 0{\cdot}00194$,
also from (2), $y_1 - y_2 = 0{\cdot}00135$.
And $y_1 - y_1 = 0$.

Adding these, $5y_1 - (y_1 + y_2 + y_3 + y_4 + y_5) = 0'{\cdot}00344$.
Equation (1) reduces this to $5y_1 = 0{\cdot}00344$.

$$\therefore\ y_1 = \frac{0'{\cdot}00344}{5} = 0'{\cdot}000688,$$

$$\begin{aligned}
y_2 &= y_1 - 0'{\cdot}00135 = -0'{\cdot}000662,\\
y_3 &= y_1 - 0{\cdot}00012 = 0{\cdot}000568,\\
y_4 &= y_1 - 0{\cdot}00003 = 0{\cdot}000658,\\
y_5 &= y_1 - 0{\cdot}08194 = -0{\cdot}001252.
\end{aligned}$$

Hence latitude $P = 44°\ 28'{\cdot}0357 + 0{\cdot}000688 = 44°\ 28'{\cdot}03639$ N.
$L = 44\ \ 13{\cdot}7212 - 0{\cdot}000662 = 44°\ 13{\cdot}72054$ N.
$N = 43\ \ 30{\cdot}4326 + 0{\cdot}000568 = 43\ \ 30{\cdot}43317$ N.
$S = 43\ \ 23{\cdot}5663 + 0{\cdot}000658 = 43\ \ 23{\cdot}56696$ N.
$F = 43\ \ 47{\cdot}4714 - 0{\cdot}00125\ \ = 43\ \ 47{\cdot}47015$ N.

which are the most probable and consistent latitudes which can be obtained from the observations by giving to each its full and proper influence.

Taking the longitude of the first meridian through *P* as - - - - - -	63° 45′·9147 W.
Difference of longitude *L*, west of *P*, - -	30·40243
Longitude of *L*, - - - - -	64° 16′·31713 W.
Difference of longitude *N*, west of *L*, - -	1 4·54546
Longitude of *N*, - - - - -	65° 20′·86259 W.
Difference of longitude *S*, west of *N*, - -	40·17836
Longitude of *S*, - - - - -	66° 1′·04095 W.
Difference of longitude *F*, west of *S*, - -	8·31413
Longitude of *F*, - - - - -	66° 9′·35508 W.

The differences of longitude are the best that can be obtained by combining all the observations from *P* to *F* inclusive, according to their values; should it be found necessary to change the longitude of *P*, the first meridian, all the others depending on it must be altered by the same amount.

CHAPTER IX.

ON PROJECTING AND SKETCHING.

BEFORE laying the observations down we must examine the paper to see if it will suit our purpose; it should be tough, not liable to tear or crease, and of uniform texture and thickness.

The sheets of paper to be used for projecting should be therefore carefully selected, the thin parts at the edges of each sheet cut off in such a manner that their edges may be straight, and making right angles with each other at the corners, leaving the remaining part of the sheet of uniform thickness throughout; the sheets should then be placed flat in covered drawers or boxes prepared for the purpose. It will be found convenient to fit these to slide in and out under the chart table.

The method of backing the paper with linen or cotton is objectionable, because after the sheet is cut from the board it contracts unequally, and distorts the plan.

Rolling up paper distorts any figure drawn on it, and the thicker the paper the greater the distortion will be.

The top of the chart table must be as nearly plane as possible, otherwise the lines drawn upon it will not be straight.

The instruments should be examined *before* they are used, and their accuracy tested; we will therefore enumerate the most important, and describe the modes of testing them.

Brass Scales.—In addition to the standard brass scale of four feet graduated to inches and parts of inches, it is convenient to have several others of different lengths between three feet and eighteen inches inclusive; these should be carefully compared with the standard when at the same temperature, using a good beam compass; when in use the brass scale should be laid flat on the projecting sheet, and at other times kept in flat boxes.

Steel Straight-edges.—A set of steel straight-edges of various lengths, from five feet to eighteen inches inclusive, are required. When in use the straight-edges must be laid flat on the chart

table, at other times they must be suspended vertically from one end. The straightness of the edges must be tested by comparing the fine pencil lines drawn along the edge when the steel is placed in reverse positions on the sheet.

Beam Compass.—Sets of these instruments with beam of various lengths, from 4½ to 1½ feet inclusive, are required. Examine the legs or pins to see that their points are fine and symmetrical with regard to the axis of the pin, which should be cylindrical with a conical point; that they sit firmly in their sockets and do not wabble in the slightest degree when the screw is set tight; that the screw for adjusting the distance between the points of the pins, when turned uniformly, gives uniform motion to the plate carrying the leg at the end of the beam, and that the screws at the top of the leg plates secure them firmly and evenly to the beam, so that when once set, the distance between the points of the legs remains invariable.

Circular Protractors.—These are very useful for laying off angles for the projection of points that are not more than an inch beyond their respective circumferences. They must be tested by comparing with each other angles of the same number of degrees laid off at different points of the circumference of the protractor, and also with angles of the same size projected by the chords.

Station Pointers.—These should be in pairs of various sizes from 12 to 3 inch radius inclusive; of each pair one should lay off the small angle to the right, and the other should lay off the small angle to the left. To examine one of these instruments, place its movable legs on opposite sides of, and as near as possible to, its fixed or zero leg; clamp the left leg firmly in that position; tighten the clamp of the right leg so as to allow a slight force to move it, and then move it gently round the circumference until it has approached the left leg as near as it possibly can, return it in the same manner to its position of nearest approach to the zero leg, and clamp it firmly there; loosen the clamp of the left leg sufficiently, and treat it in the same way. If the legs whilst moving round the circumference meet with nearly uniform resistance, they are well centred; but if, on the contrary, they or either of them meet with great resistance whilst passing along one part of the circumference, and little or none when passing along another part of it, the centering of the instrument will require adjustment. Set the right leg at reading 180°, clamp it securely, and place the instrument flat on a sheet of paper; draw a fine pencil line along the edge of the zero leg, and also along the right leg, prick through the centre of the instrument and remove it; place a steel straight-edge on the paper, so that its edge may coincide with the pencil line drawn along the zero leg. The other part of the

edge of the steel should pass through the central point, and run along the pencil line which was drawn along the edge of the right leg; if these conditions are satisfied, test the left leg in the same way, and if this is satisfactory, set the right leg to reading 90° and the left leg to reading 270°; replace the instrument on the paper in its former position, and draw fine pencil lines along the edges of the movable legs; remove the station pointer, and place the steel straight-edge on the paper so that its edge may coincide with the outer extremities of the pencil lines just drawn along the edges of the movable legs; the edge of the steel should coincide with the pencil lines through their whole length, pass through the centre point, and be perpendicular to the zero line; if this prove satisfactory, set the movable legs at readings 10° and 100° respectively, then at 20° and 110°, and so on at readings which differ from each other exactly 90°; place the instrument on the paper so that its centre coincides with the central point of the lines already drawn, and the edge of one of the movable legs coincides with one of the pencil lines: the edge of the other movable leg ought to coincide with the pencil line which is perpendicular to the pencil line which agrees with the edge of the other.

Straight-edged Protractors varying from 12 to 6 inches long inclusive are required, and should be tested by comparing the angles laid off by them with the same angles projected carefully by means of their chords.

Drawing Pencils.—These must be sufficiently hard to draw a fine line on the paper without cutting into its surface, and to keep a fine edge whilst drawing a straight line 6 feet long. When used for ruling they must be cut with thin flat edges; whilst ruling a straight line the pencil must be kept with one of its flat sides in contact with the edge of the steel, its edge pressing evenly on the paper, and its length making a small constant angle with the vertical.

The scale on which the working sheets should be projected depends upon the nature of the coast, and the fulness of detail required. In all cases it must be sufficiently large to enable the surveyor to lay down his work clearly and distinctly, which he should always do as soon as possible after the observations have been made. He must assure himself that no peculiarity worthy of notice has escaped him, and that his plan exhibits distinctly everything necessary to the safe navigation of the coast he is delineating; he must therefore examine everything with great care, and go much fuller into detail in his working sheets than will be necessary in those published from them for the use of navigation.

One inch to 1,000 or 1,500 feet will generally be sufficiently large a scale for the plane working sheets, and be within the capacity of an ordinary chart room on board a surveying vessel. In one of these scales the main triangulation, harbour, coast line, and inshore soundings should be projected; and from the plane sheets a Mercatorial projection on the scale of from 1 to 2 inches to a mile should be reduced, on which the off-shore soundings must be projected as soon as obtained.

The working sheets when not in use should be kept in a shallow covered drawer fitted to slide in and out under the chart table. In this they should be laid perfectly flat one on the other, with three or four sheets of paper, on which future projections are to be made, laid under them in juxtaposition to the working sheet they are intended to join on to, respectively; by this means the sheets attain similar conditions of dryness. This precaution is very necessary in damp changeable climates.

As soon as possible after the preliminary angles have been observed with a small theodolite at the main stations, a projection of the main stations, on a small scale, should be made with a protractor. It will be found very useful to show the relative positions of the main stations, and how the sheets for the plane scale projection should be arranged, so that the main points may lie well on them as they follow each other in succession; this requires particular attention, because a judicious arrangement of the sheets beforehand saves time, labour, and paper.

The angles of the main and secondary triangles should be projected by means of their chords, the length of which will be found conveniently from a table of natural sines, by remembering the natural sine of half an angle to radius 10 is the same length as the chord of the whole angle to radius 5. To use the tables, which are generally calculated to radius 10, divide the angle by 2, and take from the tables the natural sine corresponding to the half angle thus determined; this, as we have seen, is the same length as the chord of the whole angle to radius 5. Suppose we wish to find the chord of 32° 10′ to radius 30, take from the tables the natural sine of 16° 5′, and multiply it by 6; the result will be the length of the chord of 32° 10′ to radius 30.

When a table of natural sines is not available, a table of natural versines may be made to answer the purpose as follows. Since sine angle $A =$ versine $(90° + A)$ less radius — in the foregoing example add 90° to 16° 5′, making 106° 5′, take out the natural versine of 106° 5′, and from it subtract radius 10; this will give the natural sine of 16° 5′ to radius 10, which is equal to the chord of 32° 10′ to radius 5.

Suppose A, B, and C (Fig. 38) denote respectively the three principal stations to be projected on one of the working sheets, and that AB is the longest side of the triangle ABC, or that AB is not less than either of the other two sides. The small scale protractor plan of the main stations, recommended to be made immediately angles have been obtained at them, will point out the best position of AB on the sheet about to be projected. Lay the sheet of paper and the standard brass scale flat on the chart table, lay the steel straight-edge on the paper with its ruling edge running along the line which AB ought to occupy, and place weights along the top of the steel near its

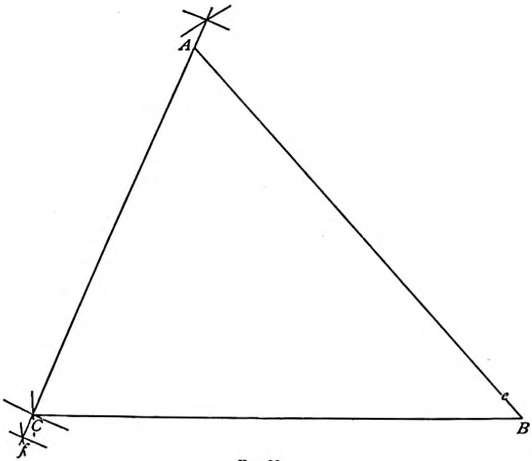


FIG. 38.

edge to ensure its retaining its place whilst ruling the straight line AB with the pencil, which should be drawn right across the sheet from margin to margin; this line must be examined carefully before removing the steel to see that it lies evenly along the edge of the steel throughout its entire length. The steel must then be turned over on its ruling edge so that the side which was previously up may lie down on the paper on the opposite side of the pencil line just ruled; bring the edge of the steel to touch the pencil line near its extremities, about an inch inside the margins of the paper respectively, and see if the edge of the steel coincides with the pencil line through its entire length, which it ought to do. Being satisfied with the

straight line AB, find from the small scale projection the position the point A ought to occupy on the line already drawn: this does not require great care, only A ought to be at least 1 inch from the margin of the sheet. Having determined the whereabouts of A, with a fine pricker held vertical prick through its position on the pencil straight line and enclose the point with a fine circular ink line which will serve to point out its place. In order to fix the mind we will suppose the scale of projection to be 1 inch to 1,500 feet; the triangulation gives the calculated length of AB in feet. From the logarithm of the number of feet in AB subtract the logarithm of 1,500, the natural number corresponding to the difference will be the number of inches from A to B on the sheet. This length must be carefully taken from the brass scale with the beam compass; this done, place the point of one leg over A and the point of the other leg over the middle of the pencil line near the position B should occupy and just clear of the paper, press the point at A gently into the small hole denoting the point A made with the pricker, and so that the point of the other leg at B may follow it and make a small indent on the middle of the pencil line to denote the position of the point B, prick this carefully through with the pricker, and ring it in with a small ink circle of the same size as that previously made at A.

In a similar manner find the lengths of AC and BC respectively in inches. With one pair of beam compasses take from the brass scale the length of AC just determined, and with another pair that of BC; this must be very carefully done, as the whole sheet will depend upon the triangle ABC. Place one leg of the first pair at A, with its other leg near the position C ought to occupy; place one leg of the second pair at B with its other leg near the position of C; round the centres A and B respectively, describe with the points of the legs near C two fine circular arcs intersecting each other; the point of intersection will be C's place on the sheet, which must be pricked in and ringed round in a manner similar to the points A and B. Join AC and BC respectively with straight pencil lines very carefully drawn and produced both ways to the margin of the sheet.

The triangle ABC must now be carefully examined. Each of the angles A, B, and C are known, and we proceed to measure them by means of their chords, to see if the projected angles agree with those given by observation, in the following manner. Let c be the number of inches in AB, b the number in AB, and d the whole number which is nearest to $\frac{b+c}{20}$. Take $10d$ for radius, and let γ be the chord of the angle A to this

radius; with a beam compass take $10d$ inches carefully off the brass scale, and from centre A with this radius describe two small circular arcs cutting AB in e, and AC produced if necessary in f, respectively; with a second beam compass take from the brass scale γ inches very accurately, and from the point e as a centre, with radius γ inches, describe a fine small circular arc which ought to pass through the point f. If this condition is not satisfied, the triangle ABC must be examined, commencing with the calculation of the sides, to discover the error that must exist somewhere, and no further progress can be made until this point has been satisfactorily settled. The angles B and C must then be tested in the same way and, proving satisfactory, the points A, B, and C may be considered well established and be those to which all the other positions subsequently projected on the sheet must be referred by means of the angles which the straight lines drawn to them from the points A, B, and C make with AB or AC, BA or BC, and CA or CB, respectively. When the distance of the point to be projected from the angular point is more than 10 inches, the method of chords should be used to determine the position of the straight line passing through it and the angular point, but when less than 10 inches a good protractor will answer the purpose.

Of the lines of reference, that should be preferred which makes the smallest angle with the line to be projected from it; and of the angular points, those nearest to the object to be projected from them. These are important considerations and particular attention must be paid to them.

After the points have been satisfactorily projected on the working sheet, the sketching and sounding sheets are taken from it as follows. Lay the paper for the sketching or sounding sheet, as the case may be, flat on the chart table, immediately under the part of the working sheet to be transferred to it, taking particular care to keep both sheets perfectly smooth and flat, place weights on the working sheet near the principal points to be placed on the sheet below it, and so as not to hide or obscure any of the other points required, and when the two sheets are firmly and securely fixed together, with a fine needle, held vertical, prick through the points on the upper sheet, which are required to be on the under sheet, taking care to press the needle through it; remove the upper sheet and with a fine sharp pencil make a small ring round each prick mark on the lower sheet, denoting its name. Draw straight pencil lines to join the principal stations, and produce them both ways to the margin of the sheet. Through the principal station near the middle of the sheet draw the true and magnetic meridians

right across the sheet; note the distance in feet between the two stations that are the furthest from each other, in order that a scale may be made from it as follows. Let i be the number of inches between the two stations, measured with a beam compass and brass scale, and d the distance in feet between the two stations; the length on the sketching sheet of 1,000 feet is $1{,}000 \times \frac{i}{d}$ inches, which must be immediately taken from the brass scale with a beam compass and applied to a straight line previously drawn on a convenient part of the

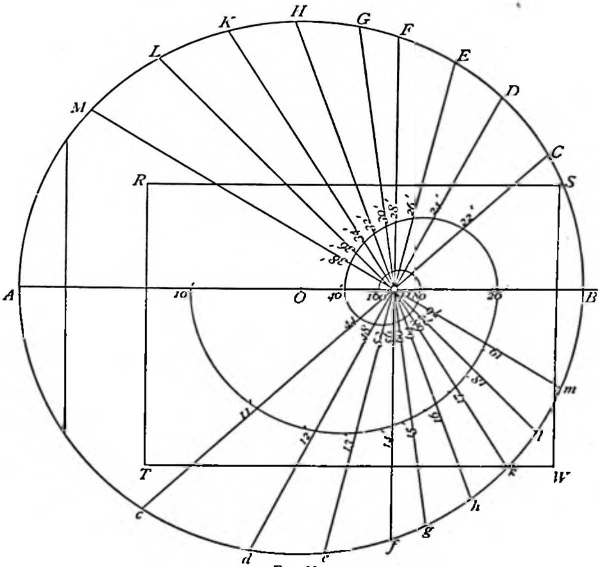


Fig. 39.

sketching sheet. Divide the length thus laid off into ten equal parts, each of which will represent the length of 100 feet on the sheet. Subdivide the first 100 feet into 10 equal parts, each of which will represent 10 feet on the sheet.

A scale showing the distances on the sheet corresponding to the vertical angles subtended by a 10 foot board is constructed as follows, and will be found very useful both in sounding and sketching.

Draw a straight line APB (Fig. 39) right across a sheet of paper 6 or 8 inches square, make AP 4 inches and PB 2 inches long respectively, bisect AB in O and from centre O, with radius OA or OB, describe the circle $AMGCBmgc$, take $PC=2{\cdot}2$ inches, $PD=2{\cdot}4$ inches, $PE=2{\cdot}6$ inches, $PF=2{\cdot}8$ inches, $PG=3$ inches, $PH=3{\cdot}2$ inches, $PK=3{\cdot}4$ inches, $PL=3{\cdot}6$ inches, and $PM=3{\cdot}8$ inches. With a ruling pencil and steel straight edge draw very carefully the straight lines MP, LP, KP, HP, GP, FP, EP, DP, and CP, and produce them respectively to meet the circumference of the circle in the points m, l, k, h, g, f, e, d, and c. Then, if AP represents the distance on a sheet at which a pole 10 feet long subtends an angle of 10 minutes, Pc will represent the distance at which the same pole will subtend an angle of eleven minutes, Pd that at which it will subtend an angle of 12 minutes, and so on to PB which corresponds to an angle of 20 minutes. This is proved as follows:

Since $PB=2$ inches, and $PC=2{\cdot}2$ inches,

$$PC=1{\cdot}1\times PB.$$

Also $$AP\times PB=PC\times Pc;$$

$$\therefore AP\times 10=11\times Pc.$$

$\therefore$ if $AP=$ distance at which pole subtends, 10′,

$Pc=$ „ same pole „ 11′,

and similarly for all the others.

A more accurate construction will be made by calculating the lengths of BC, BD, etc., and of AC, AD, etc., and from A and B as centres, with the above as radii, to describe small circular arcs which will intersect each other at right angles in the points C, D, etc., and determine their respective positions very accurately. Join OC and CB, then

$$CB=2OC\sin\frac{COB}{2}.$$

$$\text{But } 4OC\times OP\sin^2\frac{COB}{2}=\{PC+OC-OP\}\{PC+OP-OC\},$$

$$\therefore CB^2=4OC^2\sin^2\frac{COB}{2},$$

$$=\frac{OC}{OP}\{PC+OC-OP\}\{PC+OP-OC\}.$$

Now $OC=3$ inches, and $OP=1$ inch,

$$\begin{array}{ll} \therefore\ PC=2\cdot2 \text{ and } & CB^2=3\times4\cdot2\times0\cdot2 \\ PD=2\cdot4 & DB^2=3\times4\cdot4\times0\cdot4 \\ PE=2\cdot6 & EB^2=3\times4\cdot6\times0\cdot6 \\ PF=2\cdot8 & FB^2=3\times4\cdot8\times0\cdot8 \\ PG=3 & GB^2=3\times5 \\ PH=3\cdot2 & HB^2=3\times5\cdot2\times1\cdot2 \\ PK=3\cdot4 & KB^2=3\times5\cdot4\times1\cdot4 \\ PL=3\cdot6 & LB^2=3\times5\cdot6\times1\cdot6 \\ PM=3\cdot8 & MB^2=3\times5\cdot8\times1\cdot8 \end{array}$$

Again $\qquad AC = 2OC \sin \dfrac{AOC}{2} = 2OC \cos \dfrac{COB}{2},$

$$\therefore AC^2=\frac{OC}{OP}\{PC+OC+OP\}\{OC+OP-PC\}.$$

$$\begin{array}{ll} PC=2\cdot2 & \therefore\ AC^2=3\times6\cdot2\times1\cdot8 \\ PD=2\cdot4 & AD^2=3\times6\cdot4\times1\cdot6 \\ PE=2\cdot6 & AE^2=3\times6\cdot6\times1\cdot4 \\ PF=2\cdot8 & AF^2=3\times6\cdot8\times1\cdot2 \\ PG=3 & AG^2=3\times7 \\ PH=3\cdot2 & AH^2=3\times7\cdot2\times0\cdot8 \\ PK=3\cdot4 & AK^2=3\times7\cdot4\times0\cdot6 \\ PL=3\cdot6 & AL^2=3\times7\cdot6\times0\cdot4 \\ PM=3\cdot8 & AM^2=3\times7\cdot8\times0\cdot2 \end{array}$$

To calculate CB we have

log 3 - -	0·477121	
log 4·2 - -	0·623249	
log 0·2 - -	$\bar{1}$·301030	
log CB^2	0·401400	
$\therefore$ CB - -	0·200700	$\therefore$ $CB=1{\cdot}587$ in.

For AC we have

log 3 - -	0·477121	
log 6·2 - -	0·792392	
log 1·8 - -	0·255273	
log AC^2 -	1·524786	
log AC -	0·762393	$\therefore$ $AC=5{\cdot}786$ in.

Similarly	$DB = 2{\cdot}298$ inches,	$DA = 5{\cdot}543$ inches.
	$EB = 2{\cdot}878$ "	$EA = 5{\cdot}265$ "
	$FB = 3{\cdot}394$ "	$FA = 4{\cdot}948$ "
	$GB = 3{\cdot}873$ "	$GA = 4{\cdot}583$ "
	$HB = 4{\cdot}327$ "	$HA = 4{\cdot}157$ "
	$KB = 4{\cdot}763$ "	$KA = 3{\cdot}650$ "
	$LB = 5{\cdot}185$ "	$LA = 3{\cdot}020$ "
	$MB = 5{\cdot}597$ "	$MA = 2{\cdot}163$ "

Observing that the angles AMB, ALB, etc., are all right angles, because they are in the semicircle $AML \dots CB$, an ordinary traverse table may be used to determine the longer side of the two containing the right angle when the shorter side is known. Thus, if AB be taken for the distance and BC for the departure, AC will be the difference of latitude, $AB = 6$ inches, and $BC = 1{\cdot}587$ inches.

Looking in the traverse table under distance 60 we see for

difference latitude 15·5; departure = 58·0;
and for difference latitude 16·5; departure = 57·7.

Therefore an increase of 1 in the difference of latitude gives a decrease of 0·3 to the departure; consequently an increase of 0·37 in the difference of latitude will give a decrease of 0·111 to the departure, and for difference of latitude 15·87 we shall have the departure = 57·89, and therefore $AC = 5{\cdot}789$ inches. The calculated length is, as we have seen, 5·786, which shows the traverse table value is sufficiently accurate.

The points c, d, ... l, and m will be more accurately determined from the lengths of Ac, Bc, ... Am, Bm, which can be quickly and easily calculated from those of CB, CA, ... MB, MA, and the triangles AcP and PBC being similar, we have

$$Ac : AP = CB : PC;$$

$$\therefore \quad Ac = \frac{AP}{PC} \times CB$$

$$= \frac{2}{1{\cdot}1} \times CB = 2{\cdot}885 \text{ inches.}$$

Similarly $$Ad = \frac{2}{1{\cdot}2} \times DB = 3{\cdot}83 \quad ''$$

$$Ae = \frac{2}{1{\cdot}3} \times EB = 4{\cdot}428 \quad ''$$

etc., etc.

In like manner $Bm = \frac{1}{1\cdot 9} AM = 1\cdot 139$ inches,

$$Bl = \frac{1}{1\cdot 8} AL = 1\cdot 678 \quad ,,$$

$$Bk = \frac{1}{1\cdot 7} AK = 2\cdot 147 \quad ,,$$

etc., etc.

The points c and d will be best determined from Ac and Ad, whilst the points $e, f, g, \ldots m$ will be best found by using $Be, Bf, \ldots Bl, Bm$ as radii from B as a centre; the circular arcs thus described will cut the circumference $BmgeA$ at good angles respectively in the points $c, f, \ldots l$ and m; with a steel straight edge and a ruling pencil draw the straight lines MPm, LPl, etc., taking care that they pass very exactly through the point P.

The length on the sketching sheet, corresponding to the distance a pole ten feet long is from the observer when it subtends an angle of ten minutes, is $\frac{5i}{d}$ cot $5'$ inches; with a beam compass take this length carefully from the brass scale, place the point of one leg of the compass just over the point P, and that of the other over the straight line PA, and as close to the paper as possible without touching it; press one gently into the point P, so that the other may make a slight puncture on PA. Round the point thus defined draw a fine pencil line, and write $10'$ either just above or below PA; from $P \ldots PB$ lay off in a similar manner $P20' = \frac{P10'}{2}$; bisect the straight line $10'P20'$ in the point $40'$; $20'P40'$ in the point $80'$; $40'P80'$ in the point $160'$, and so on; from the point $40'$ as a centre, with a radius $40'\ 20'$ or $40'\ 10'$, describe *below* APB the semicircle $10'\ 11'\ 12' \ldots 19'\ 20'$ cutting the straight lines $Pc, Pd \ldots Pl$ and Pm in the points $11', 12' \ldots 18'$ and $19'$ respectively; from point $80'$ as a centre with radius $80'\ 20'$ or $80'\ 40'$, describe *above* the straight line APB the semicircle $20'\ 22' \ldots 38'\ 40'$ cutting the straight lines $PC, PB \ldots PL$ and PM respectively in the points $22'\ 24' \ldots 36'$ and $38'$; from the point $160'$ as a centre with radius $160'\ 40'$ or $160'\ 80'$ describe *below* APB the semicircle $40'\ 44'\ 48' \ldots 72'\ 76'\ 80'$ cutting the straight lines $Pc, Pd \ldots Pl$ and Pm in the points $44'\ 48' \ldots 72'\ 76'$; proceeding in this manner semicircles are described alternately above and below APB as far as necessary.

The oblong part $RSWT$ is then transferred to a piece of paper stretched on a small light board, which will be found

a very convenient scale to use for laying off the distances given by a ten-foot pole when sketching. Observing that $P10$ is the distance corresponding to 10′, $P11'$ to 11′, $P22'$ to 22′, etc. If the angle subtended by the ten foot pole is intermediate in size between those corresponding to two consecutive lines drawn from P, the outer leg of the compass must be placed on the circumference of the semicircle between the two lines, dividing it by the eye in proportion to the differences between the observed angle and the two corresponding to the lines between which it is intermediate in size, as well as to the lengths of the arcs of the semi-circumference between the two lines and those next adjacent to them on each side respectively.

The following are required for sketching: 1 10-inch straight-edge protractor, 1 steel straight-edge about 18 inches, 1 pair of dividers, 1 prismatic compass, 1 Sir Howard Douglas reflecting protractor, a set of small station pointers with high centering, a measuring tape, several small pieces of tracing paper cut square in a flat case for independent projections, pen-knife, india-rubber, 1 Rochon micrometer telescope, or other instrument for measuring small angles accurately, 1 sextant, with good telescope capable of reading 3° off the arc, a small theodolite, 1 aneroid barometer, 1 pole with ten feet accurately and distinctly marked upon it, etc. The small articles should be neatly fitted into a small box or case to be carried by an attendant, so that immediately after use each may be replaced in the case, a precaution which ought never to be neglected.

All the vertical heights of the stations should be distinctly written in figures on the sketching sheet; to the nearest of these the sketcher must refer the heights of the different objects as he sketches them, as well as the contour lines of equal elevation.

Thus prepared the sketcher starts from a fixed position, projecting his course accurately on the sheet as he moves along, using his traverse as a base from which to fix and sketch in the objects near him, delineating the lines of equal elevation, the watercourses and other characteristics of the country; his sketch should be bounded by the green line to seaward, and extend inland sufficiently to show the highlands and hills which are seen by vessels approaching the shore. The country coast roads generally afford a good path along which the sketcher can take his course. When in sight of a sufficient number of fixed points his position must be projected from those nearest to him; but when, as frequently happens in wooded countries, he loses sight of the points fixed on his board, he must depend upon the 10 foot pole and prismatic compass. When this

happens he will find it the better plan to project his course on a separate piece of paper on which he also sketches in the features of the country, until he again comes in sight of known points on his board, from which he can determine his place, and then, by comparing his traverse with the two fixed positions on the sketching sheet at its commencement and at its end, correct the traverse and sketch in the following manner.

Let A (Fig. 40) be the last point the sketcher is able to fix before losing sight of the known objects marked on his board; AbB, an accurate line of direction observed from A, the dis-

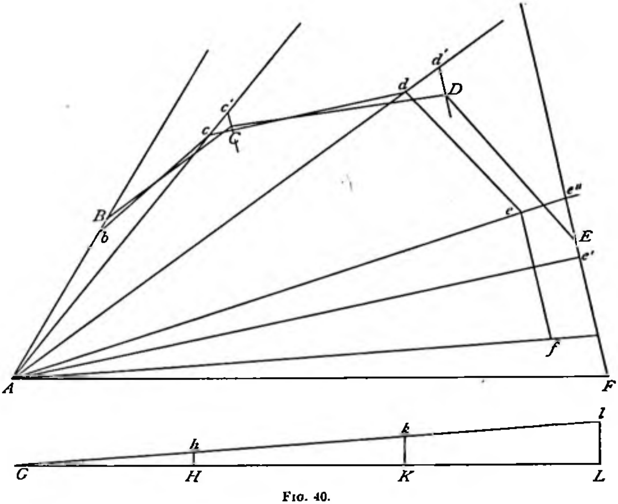


FIG. 40.

tance Ab being given by the angle of elevation of a 10 foot pole. The sketcher then proceeds from b along the lines bc, cd, de, and ef, using the pole and prismatic compass; at f he comes in sight of known objects and projects the point f and the straight line fe on his sketching board, which, when compared with the point A and the straight line AbB on the paper, gives F for the correct position of f, and $Fe'Ee''$ for that of fe. The error here is purposely exaggerated in order to show clearly the mode of correcting the positions of b, c, d, and e.

Make $Fe'=fe$; join Ae', Ae, Ad, and Ac, which produce in-

definitely, and make $Ae''=Ae'$; with proportional compasses make $\frac{AB}{Ab}$, $\frac{Ac'}{Ac}$, $\frac{Ad'}{Ad}$, and $\frac{FE}{Fe}$, all equal to $\frac{Ae''}{Ae}$, thus defining the points B, c', d', and E; join $e''E$, and through the points c' and d' draw $c'C$ and $d'D$ respectively parallel to $e''E$. Draw the straight line GL (Fig. 40) across the paper and make $GH=bc$, $HK=cd$, and $KL=de$, through the points H, K, and L draw Hh, Kk, and Ll, perpendiculars to GL, make $Ll=Ee''$, join Gl cutting Hh in h and Kk in k, make $d'D=Kk$ and $c'C=Hh$, then B, C, D, and E will be the correct positions of b, c, d, and e respectively. The points A and F are projected on the sketching board, the points B, C, and D must be taken off the auxiliary sheet and placed on the sketching sheet with respect to A and F; by means of the proportional compasses the part of the sketch between A and b on the paper is drawn along AB on the sketching sheet, that between b and c is sketched in between B and C on the sketching sheet, and so on.

However, when the traverse with pole and compass is carefully made, the error in position of the last point, f, and the direction of fe differ generally very slightly from the truth.

When the sketcher is about to lay down a line of direction to any object, he should select from the fixed points on his sketching sheet that which is nearest to the object he is about to project, if the angle between the two objects is small and they differ much in elevation. When he is using a reflecting instrument he must select a distant well-defined object which makes a right angle, or as near thereto as possible, with the line bisecting the angle between the two objects, and observe the angles between it and each of them. The difference between the two angles will give the small horizontal angle between the two objects he wishes to obtain with sufficient accuracy.

The heights in feet above the high-water line of the fixed positions on the board, or as many as possible, are written on the sketching sheet; the others as he fixes them, the sketcher must determine, whenever he possibly can, differentially from the nearest known elevation, or when the vessel is in sight and sufficiently near, from the height of her truck above a fixed mark on the mast just above the gunwale in the following manner. With a theodolite he observes the readings of the vertical circle when the horizontal wire of the telescope passes through the truck and the fixed line on her mast respectively, and when the telescope bubble is placed in its central position.

Let h be the height of the ship's truck above the mark on the mast,
a the height of the mark above the water,
T the tide pole reading of the high line,
t the tide pole reading at the time of observation,
d the height of the telescope above the ground,
c the correction due to the curvature of the earth,
all in feet.

Also let α be the reading of the vertical circle when the horizontal wire of the telescope passes through the truck.

Let β be the reading corresponding to the bubble, and γ that for the water line—all in degrees, minutes, and seconds.

Then the height of the ground on which the theodolite stands above the high line in feet will be

$$=h\frac{\tan(\beta-\gamma)}{\tan(\alpha-\gamma)}+a+t-T-d-c, \dots\dots\dots\dots\dots\dots (1)$$

When, which is generally the case, $\beta-\gamma$ and $\alpha-\gamma$ are small, the expression reduces itself to

$$h\frac{\beta-\gamma}{\alpha-\gamma}+a+t-T-d-c \dots\dots\dots\dots\dots\dots (2)$$

The tide observations give $T-t$; c is generally small, and in such cases may be considered to vary inversely as $(\alpha-\gamma)^2$.

When $h=120$, which is a fair average height, and $\alpha-\gamma=30'$, which is as far as the vessel should be from the observer when this method is used, c will equal 4 feet.

Then, for $\alpha-\gamma=33'$, $c=\frac{4}{(1{\cdot}1)^2}=3{\cdot}3$ feet.

$=36'$, $=\frac{4}{(1{\cdot}2)^2}=2{\cdot}8$ „

$=39$, $=\frac{4}{(1{\cdot}3)^2}=2{\cdot}4$ „

$=42$, $=\frac{4}{(1{\cdot}4)^2}=2{\cdot}0$ „

$=45$, $=\frac{4}{(1{\cdot}5)^2}=1{\cdot}8$ „

$=48$, $=\frac{4}{(1{\cdot}6)^2}=1{\cdot}6$ „

$=51$, $=\frac{4}{(1{\cdot}7)^2}=1{\cdot}4$ „

$$\alpha-\gamma=54, \quad c=\frac{4}{(1\cdot8)^2}=1\cdot2 \text{ feet.}$$
$$=57, \quad =\frac{4}{(1\cdot9)^2}=1\cdot1 \quad ,,$$
$$=60, \quad =\frac{4}{(2)^2}=1 \quad ,,$$

When the height of the mast changes, c will vary with h^2; therefore, when the height differs from 120 feet,

$$\text{Correction}=c\left(\frac{h}{120}\right)^2=c\left(1+\frac{h-120}{120}\right)^2$$
$$=c\left(1+\frac{h-120}{60}\right) \text{ very nearly when } h-120 \text{ is small}$$
$$=c\frac{h-60}{60}, \text{ when } h \text{ does not differ much from 120 feet.}$$

Example. $\alpha-\gamma=40'$, and $h=100$ feet.

$$\text{Correction}=2\cdot3\times\frac{40}{60}=2\cdot3\times\frac{2}{3}=1\cdot5 \text{ feet.}$$

Take another example, where $h=127\cdot5$ feet, $a=14\cdot5$, $d=4\cdot5$, $T=30, t=11, \beta=0'\,30''$, $\gamma=-(1^\circ\,15'\,20'')$, and $\alpha=-(0^\circ\,35'\,15'')$.

$$\therefore \beta-\gamma=1^\circ\,15'\,50''=75'\cdot8 \text{ approximately,}$$
$$\alpha-\gamma=\quad 40 \quad 5 \ =40\cdot1 \quad ,,$$

$$\text{Curvature corr.}=2\cdot3\times\frac{67\cdot5}{60}=2\cdot3\times\frac{4\cdot3}{4}=2\cdot5 \text{ feet.}$$
$$\therefore a+t=25\cdot5 \text{ feet,} \quad T+d+c=37 \text{ feet.}$$
$$\text{Height of ground}=\frac{75\cdot8}{40\cdot1}\times127\cdot5+25\cdot5-37=230 \text{ feet.}$$

When the station or object is not very far from the theodolite the difference of elevation of the two places can be found by standing the measuring pole upright on the ground on which the object is situated, and noting the reading of the vertical circle when the horizontal wire of the telescope passes through the top and bottom mark defining the 10 feet, and also when the bubble stands in its central position; the readings being α, β, and γ, as before, the difference of elevation will be $=\frac{\beta-\gamma}{\alpha-\gamma}10-d$.

Suppose the ground on which the theodolite stands to be 247 feet above the high line, its telescope being 4 feet above the ground, and the bottom mark of the pole being 2 feet above

the ground it stands on; the reading of the vertical circle gave $a = -45' \ 20''$, $\gamma = -(1^\circ \ 2' \ 30'')$, $\beta = -0' \ 30''$. The ground on which the theodolite stands is therefore $\frac{62}{17 \cdot 2} \times 10 - 4 + 2 = 34$ feet higher than the ground on which the pole stands. The pole ground is therefore $247 - 34 = 213$ feet above the high line.

When using the 10 feet pole during sunshine with a low altitude of the sun, the pole must not be placed between the observer and the sun, but the angle at the observer between the pole and the sun should not be less than a right angle; the face of the pole carrying the marks defining the 10 feet should be so held that its normal bisects the angle at the pole between the observer and the sun; this enables the observer to see the marks on the faces of the pole distinctly.

Example. The telescope of a theodolite is 251 feet above the high line, a 10 feet pole held vertically with its bottom mark 2 feet above the ground was observed, and the readings of the vertical circle gave bubble reading $-0' \ 30''$; top mark of pole $-(45' \ 20'')$; bottom mark of pole $-(1^\circ \ 2' \ 30'')$.

Height of telescope above the ground on which the pole stands is therefore $\frac{62}{17 \cdot 2} \times 10$ ft. $+ 2$ ft. $= 38$ feet, and the height of the ground on which the pole stands 213 feet above the high line.

When the sketcher has projected his own position and that of another object near it on his sheet, the differences between heights of the telescope of his theodolite and that of the other point can easily and quickly be determined from the distance on the sheet between the two points, measured with a pair of dividers, as follows.

Let h be the difference between the heights required in feet; a, β, the readings of the vertical circle of the theodolite as before; n the length in inches on the sheet between the two points in the sketching sheet measured with a pair of dividers and the scale of inches; s the number of feet represented by 1 inch on the sketching sheet.

Then we have
$$h = d \tan(a - \beta)$$
$$= d(a - \beta)$$
when $\overline{a - \beta}$ is small and expressed in circular measure;
$$\therefore \quad h = \frac{d}{6} \times \frac{a' - \beta'}{573} \text{ very approximately}$$
when a' and β' are expressed in minutes of angle.

Since $d = sn$, substituting it for d in the above,

$$h = \frac{s}{6 \times 573} n(\alpha - \beta)' \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

When the scale of construction is determined, $\frac{s}{6 \times 573}$ can be calculated, and will be constant for all the sketching sheets of the survey.

For example, suppose $s = 1500$,

$$\frac{s}{6 \times 573} = \frac{1500}{6 \times 573} = \frac{250}{573} = 0{\cdot}43455,$$

and $h = 0{\cdot}43455 \times n \times (\alpha - \beta)'$
$= 0{\cdot}43455 \times$ length on sheet in inches $\times$ minutes of angle.

Suppose, for example, $\alpha = 1° \; 5' \; 20''$, $\beta = 40''$, and $n = 2{\cdot}414$ in.,

then $h = 0{\cdot}43455 \times 2{\cdot}414 \times 64{\cdot}67 = 60$ feet.

When neither of the foregoing methods are available the aneroid barometer can be used differentially for the same purpose with good effect. The principle upon which this rests may be explained as follows. Suppose A and B be two places, of which B is the higher; draw the straight line Bb horizontal through B, meeting the vertical straight line Ab through A in b; then Ab is the excess of the height of B above the high line over that of A.

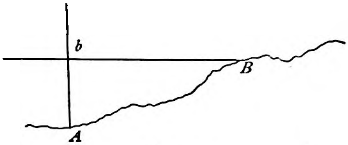


Fig. 41.

The pressures of the atmosphere in the horizontal line Bb are supposed to be equal throughout; let this pressure be expressed by some quantity P_2. The pressure of the atmosphere at A is equal to the pressure at b together with the weight of the column of air between b and A, whose height $Ab = h$ feet supposed; let w be the weight of a volume of air standing on a unit of base one foot high when at temperature 32° Fahrenheit, and under a unit of pressure; t = half sum of the temperatures at b and A less 32° F., and P_1 the pressure of the atmosphere at A.

It has been found by experiment that air under a constant pressure expands 0·00208 times its volume for every degree Fahrenheit its temperature is increased. The weight of a column of air in Ab standing on a unit of base at A may be

taken equal to $\frac{hw(P_1+P_2)}{2(1+0\cdot00208t)}$ with sufficient accuracy for our purpose.

$$\therefore P_1 = P_2 + \frac{hw(P_1+P_2)}{2(1+0\cdot00208t)},$$

$$\therefore h = \frac{2(P_1-P_2)}{P_1+P_2} \cdot \frac{1+0\cdot00208t}{w}.$$

If, therefore, a_1 and a_2 be the aneroid readings at A and B respectively, they will have the same ratio to each other that P_1 and P_2 have, and therefore

$$h = \frac{2(a_1-a_2)}{a_1+a_2} \cdot \frac{1+0\cdot00208t}{w}.$$

Now w varies with the latitude of the place in such a manner that if l be the latitude for this value, and W the value of w when the latitude is 45°,

$$w = W(1-0\cdot00256 \cos 2l).$$

Comparing the values of W given by various observations I have adopted 26,400 feet for the value of $\frac{1}{W}$, and from it have constructed the following table for latitude 45°, which will be found useful.

Half Sum of the Aneroid Readings.	Difference in Height corresponding to *one* inch difference in the Aneroid Readings. Temperature Fahrenheit.			
	2°	32°	62°	92°
30 inches.	825·1 feet.	880·0 feet.	934·9 feet.	989·8 feet.
29 "	853·5 "	910·3 "	967·1 "	1024·0 "
28 "	884·0 "	942·8 "	1002·0 "	1060·5 "
27 "	916·8 "	977·8 "	1039·0 "	1100·0 "
26 "	952·0 "	1015·4 "	1079·0 "	1142·0 "
25 "	990·1 "	1056·0 "	1122·0 "	1187·8 "
24 "	1031·4 "	1100·0 "	1168·3 "	1237·3 "
23 "	1076·2 "	1148·0 "	1219·5 "	1291·0 "
22 "	1125·0 "	1200·0 "	1274·9 "	1349·8 "
21 "	1179·0 "	1257·0 "	1335·6 "	1414·0 "
20 "	1235·7 "	1320·0 "	1402·4 "	1484·7 "

For latitude l each of the differences in height in the table must be multiplied by $\frac{1}{1-0\cdot00256 \cos 2l}$.

the value of which when $l=0°$ is $=1{\cdot}0025$,
$l=15°$ is $=1{\cdot}0022$,
$l=30$ is $=1{\cdot}0013$,
$l=60$ is $=0{\cdot}9987$,
$l=75$ is $=0{\cdot}9974$.

The aneroid should always be kept in the shade held with its face upwards, and dial plate horizontal and at the same distance from the ground when making the observation; the eye being always similarly situated with respect to the index whilst reading off. The thermometer reading and the time must always be taken and noted with the aneroid reading.

Take the following example: In latitude 45° nearly, at a station whose height above the high line is 245 feet, the reading of the aneroid was 29·625 inches, the temperature 60° Fahr., and time 10^h A.M.; at 10^h 26^m A.M. at position *b* on the sketching sheet the aneroid reading was 29·382 inches, and temperature 59° Fahr.; at position *c* on the sketch at 10^h 58^m A.M. the aneroid reading was 29·573 inches, and temperature 61° Fahr.; at station *D*, 740 feet above the high line, the aneroid reading at 11^h 54^m A.M. was 29·207 inches, and temperature 60° Fahr. The hourly observations on the barometer on board the ship on the same day, which was far from the sketcher, showed that the barometer from 10^h A.M. to noon rose steadily 0·015 inches. Supposing the pressures of the atmosphere at the places where the aneroid readings were taken to vary similarly to those observed on board, we bring all the pressures to what they probably would have been at 10^h A.M., and correct the aneroid accordingly. We therefore subtract 0·014 from the aneroid reading at *D*, 0·007 from that taken at *c*, and 0·003 from that observed at *b*, reducing the *D* reading to 29·193 inches, the *c* reading to 29·566 inches, and the *b* reading to 29·379 inches. The difference between the aneroid readings at *a* and *D* thus corrected is 0·432 inch, their half sum 29·4 inches, and the mean temperature 60° Fahr. Entering the table with these values, we find 1 inch difference in the aneroid readings gives 950 feet, and consequently 0·432 inch gives 410 feet; this should be equal to the difference in the height of the two stations *D* and *a* above the high water line, subtracting 245 feet, *a*'s height, from 740 feet, *D*'s height, we find the differences in their heights by direct observation to be 395 feet, or 15 feet less than that given by the aneroid; therefore from the heights of *b* and *c* determined from the aneroid readings, in the same way, 3·7 per cent. must be subtracted.

The difference between the reading at *a* and that at *b*

corrected to 10^h A.M. is 0·246 inch, with a mean temperature of $59\frac{1}{2}°$ Fahr., the half sum of the aneroid readings being 29·5 inches. The table gives for these quantities 946 feet for 1 inch difference in the aneroid readings; therefore 0·246 inch gives 233 feet as the difference between the heights of *a* and *b*; subtracting 9 feet, which is 3·7 per cent. on 233, we obtain 224 for the correct difference in height between the two stations; therefore adding this to 245 feet, the height of *a*, we find *b*'s height to be 469 feet; in the same manner we find *c* to be 53 feet higher than *a*, or 298 feet above the high water line.

After each day's work in the field the sketcher should completely finish and ink in all he has obtained before commencing work on the following day, and when the sheet as far as the line of junction of the next succeeding sheet, which must be clearly drawn, is full, it must be taken off the board and laid flat in a drawer reserved for the purpose.

The sketch is at convenient times copied into the large working sheets, but the sketches must always be kept as the authentic record of the work, to be referred to when necessary.

The methods by which the sketcher can determine his position on the sheet by means of two angles between three known points on his sheet are fully described in Chapter XIII., on Sounding, to which the reader is referred.

CHAPTER X.

TIDES.

THE tidal fluctuations in the surface of the sea, except in places where the rise and fall is very small, are apparent to all who visit the sea coast. Such persons cannot fail to notice the varied state of the beach at different times, and to see there a well-marked line—called the high line—beyond which the sea seldom or never rises, but below which they will observe the water line in various positions, sometimes approaching the high line so close as to leave only a narrow strip of the beach uncovered, at others receding so far from the high line as to leave a large extent of the beach exposed to view.

When a surveyor is about to delineate such a coast, he must make careful observations on the tide, and reduce all the soundings he takes to some well-defined determinate level.

A few observations made to determine the times and heights of high and low water generally show that high and low water follow each other in regular order. When the time of high water is compared with the time of the moon's meridian passage, which immediately precedes it, the difference—called the luni-tidal interval—does not vary much. When the ranges of the tides which happen about full and new moon are compared with each other, it is found that the larger the moon's diameter is, the greater is the range of the tide. These circumstances lead us to infer that the moon and the tides are in some way related.

In order to discover what this relation most probably is, we will first consider what effect the moon's attraction is likely to produce on the ocean.

We know that the moon attracts each particle of the earth with a force which varies inversely as the square of her distance from it; consequently the nearer a particle is to the moon, the greater is the force of her attraction on it, and the difference between the moon's attracting force on a particle of the earth which is nearer to her than the centre of gravity of the earth and her attracting force on a particle at the centre, produces a

relative attracting force on the particle towards the moon; whilst on particles of the earth which are farther from the moon than the centre of gravity of the earth, there will be a relative attracting force acting from the moon. In other words, particles of the earth which have the moon above their horizon, are relatively to the centre of the earth drawn towards the moon, and those which have the moon below their horizon may be * described as relatively drawn from her. This effect is called the moon's disturbing force, which, acting on the particles of water in the ocean is the principal cause of the tides.

When describing the general effect of the moon's disturbing force, it is better to resolve it into two forces—one vertical and the other horizontal.

The vertical part of the force acts against gravity, and slightly reduces the weight of a particle of water, but its effect on the tide-wave is so slight that it will not be necessary to notice it here.

The horizontal part of the moon's disturbing force varies as the sine of twice the altitude of the moon above the particle's horizon, by which she draws it horizontally towards her; at the same time the particles which have the moon below their horizon are drawn horizontally towards the point on the earth's surface which has the moon in its nadir, by a force which varies as the sine of twice the arc of the earth's surface intercepted between this point and the particle. Consequently, if we suppose the moon's disturbing force to commence acting on a free ocean at rest, the particles of water which have the moon above their horizon will begin to move towards the point on the earth's surface which has the moon in its zenith; whilst the particles of water which have the moon below their horizon will move towards the point on the earth's surface which has the moon in its nadir. If the moon always remained in the zenith and nadir of the same points, the water at these points would rise above the levels they would have in an undisturbed ocean at rest, whilst the water at the places having the moon in their horizon would fall below that level; but the earth's rotation, combined with the orbital motions of the earth and moon, keep these points constantly changing, and thereby entirely alter this effect. To explain what really does happen we will premise the following.

1. A force acting on a particle at rest, and free to move, causes it to move with a gradually increasing velocity in the direction in which the force acts, so that when the force ceases to act the particle will have attained its greatest velocity.

* This is not strictly true, but very nearly so, and quite sufficient for the purpose of a general explanation.

2. If a force begins to act on a particle moving in the same direction as that in which it acts, the particle's velocity will be continually increased; but if the force begins to act on a particle moving in a direction opposite to that in which it acts, the particle's velocity will be continually diminished until it is brought to rest, after which it will move with a gradually increasing velocity in the direction in which the force acts.

3. If a force begins to act on a particle moving at right angles to its direction, it alters the direction without changing the velocity of the particle. If the direction of the particle's motion makes an acute angle with that of the force, its velocity is increased and the direction of its motion altered; if the angle is obtuse, the velocity of the particle will be diminished and the direction of its motion changed.

4. When the water in the ocean is running with a variable velocity, at the places where the velocity is the least the surface will stand highest, and where the velocity is the greatest the surface will be lowest.

5. The friction between the particles of water introduces a force resisting their motion; so that a particle of water at rest, upon which a force increasing from zero acts, will not begin to move until the force has become sufficiently large to overcome the friction; also, when a gradually diminishing force becomes so small as to equal the force of friction, it cannot increase the particle's velocity any longer, and the particle will then have attained its greatest velocity.

We are now prepared to describe generally the effect which the moon's disturbing force has on the ocean.

First, we will suppose the moon to be on the equator, and examine the effect of her action on the particles of water in the ocean near the earth's equator; a particle of water so situated, and above whose horizon the moon is rising, will be pulled by her towards the east with a force which, when resolved horizontally, varies as the sine of twice the moon's altitude; and supposing the particle at moonrise to be moving towards the west, in consequence of the moon's disturbing force before moonrise acting in that direction, the disturbing force now begins to act to the eastward, commencing from zero, gradually increases as the moon rises, and acting with friction, will continually diminish the particle's westerly velocity and bring it to rest at some time between moonrise and her meridian passage. After being thus brought to rest, the particle—still under the influence of the easterly force—will commence moving to the east with a gradually increasing velocity; after the moon has reached an altitude of 45° her horizontal disturbing force diminishes as her altitude increases until it becomes zero

on her reaching the particle's zenith and crossing its meridian; consequently at some short time before the moon crosses the meridian, her horizontal disturbing force will have so diminished as just to balance friction, and at this instant the particle will have attained its greatest easterly velocity. After passing the meridian the moon's westerly disturbing force comes into play, and acting with friction will bring the particle to rest at some time before the moon sets below its horizon; the particle will then begin to move towards the west with a gradually increasing velocity, until shortly before moonset her westerly disturbing force will have become so small as not to exceed the friction, when the particle's westerly velocity will have reached a maximum and begin slowly to diminish.

From moonset to her inferior transit the disturbing force will again act to the eastward in a manner similar to that between moonrise and her superior transit; the particle's westerly velocity will therefore gradually diminish after moonset, and coming to rest at some time between moonset and her inferior transit will afterwards move towards the east with a gradually increasing velocity which will attain a second maximum some short time before the moon's inferior transit. Between the moon's inferior transit and moonrise, her disturbing force again acts to the westward in a manner similar to that between her superior transit and moonset, producing a like effect.

Hence the particles of water in a free ocean, near the earth's equator, will when the moon is on the equator be at rest four times during a lunar day, viz., once between moonrise and her superior transit, once between her superior transit and moonset, a third time between moonset and her inferior transit, and the fourth time between the moon's inferior transit and moonrise.

The particles will also have four periods of maximum velocity during a lunar day, two when moving east, which will be at some short time before the moon's superior and inferior transits, and two when moving west, which will be at some short time before moonrise and moonset respectively.

The particles of water all round an equatorial belt of a free ocean when the moon is on the equator will consequently be in various states of motion as the earth revolves—some being at rest, others moving east with various velocities, and others towards the west in a similar manner; at those places where the particles are at rest the surface will stand highest, and at those places where the particles are moving fastest, the surface of the ocean will have fallen to its lowest point. Consequently at these places there will be four high waters and four low waters during a lunar day; the differences between the times

and levels being all equal; and thus four equal and similar lunar waves will be produced during a lunar day.

In considering the effect which the moon when on the equator has on the particles of water in a free ocean situated at a considerable distance from the earth's equator on either side of it, we will premise the following.

A particle P (Fig. 42), moving in a direction PA, is acted on by a force in direction PF, making an obtuse angle APF with

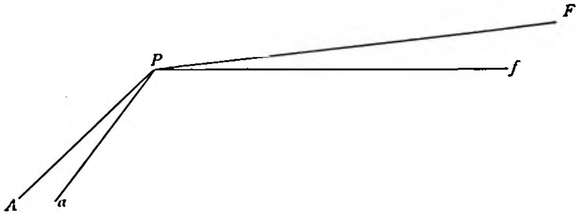


FIG. 42.

PA; the velocity of P will therefore be diminished by the action of the force, and its direction altered to another such as Pa somewhere *between* PA and PF; consequently the angle FPa will be nearer to a right angle than FPA, and the angle between the direction of P's velocity and that of the force will gradually diminish to a right angle and afterwards become acute, supposing the force to continue its action sufficiently long; if P has not been previously brought to rest, when the angle between the two directions becomes a right angle, the diminution of P's velocity will cease, and the velocity will have reached a minimum; because the instant the angle FPa becomes acute the force will tend to increase the velocity of P.

Suppose next that the force, instead of acting continuously in the direction PF gradually changes its direction, so that whilst the direction of P's velocity changed from PA to Pa, that of the force changed from PF to Pf, the only difference between this and the former case is that the angle aPf will become a right angle sooner, and therefore P's velocity reaches its minimum value earlier than if the force always kept the direction PF; also the angle fPa diminishing more rapidly than before, the directions Pf and Pa will coincide sooner; but after coincidence Pf, still continuing to alter in the same direction, will cross Pa, and take up a position on the opposite side of it, as in Fig. 43. Consequently the direction in which Pa had been revolving about P before Pf coincided with it, will cease at coincidence; and after Pf has passed it, will change and

follow PF, but slowly at first, so that the angle fPa will increase until the angular velocity of Pa round P becomes equal to that of Pf; if, therefore, the angular velocity of Pf is sufficiently great to gain more than a right angle on Pa, and cause the angle fPa to become obtuse, the velocity of P will pass through a maximum, when the angle fPa passes through a right angle and becomes obtuse.

We will next consider the motion of the particles of water situated in a parallel of latitude ten degrees north or south of the equator; the moon still being on the equator, the effect of her disturbing force will be the same in the same degree of latitude either north or south; but to fix the mind we will take a particle in a free ocean ten degrees north of the equator; under the above circumstances the moon's true bearing at rising will be east.

One hour	after	moonrise	it will be	S. $87\frac{1}{3}$°E.
Two hours	„		„	S. 84° E.
Three „	„		„	S. 80 E.
Four „	„		„	S. 73 E.
Five „	„		„	S. 57 E.
Six „	„		„	South,

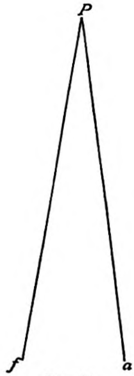


FIG. 43.

and reckoning from moonset we shall have the same number of degrees in the true bearing, only we must replace east by west, leaving south to remain.

In this case, therefore, for the first three hours after moonrise her disturbing force acts in directions nearly east, for it only changes gradually in that three hours from east to S. 80° E., whilst in the same time the magnitude of the force has changed from zero to about its maximum value; similarly, during the three hours immediately preceding moonset she draws the particles of water in directions nearly west.

During the interval between three and five hours after moonrise, and from five to three hours before moonset respectively, her true bearing only changes 23°, which is less than would be given during the same intervals by her mean rate of change of bearing; the disturbing force also during these intervals is large, and consequently a more rapid change in the direction of the particle's velocity will occur in them than during the two former.

During the interval between one hour before the moon crosses the particle's meridian, and one hour after she has done so, her true bearing changes 114°, whilst the moon's horizontal

disturbing force passes through a minimum rate when she crosses the meridian.

The foregoing remarks on the moon's motion in azimuth and the corresponding magnitudes of her disturbing force require particular attention, because the clearer our conception of the changes in the magnitudes and directions of the moon's disturbing force during a lunar day, the better the changes it induces on the motion of the particles of water during the same period will be understood.

At moonrise her disturbing force, which had previously caused the particles of water to move in a direction to the southward of west, ceases, leaving them moving in that direction, and commences to act towards the east; consequently the angle between the direction of the disturbing force, and the direction in which the particles of water are moving will then be obtuse; the disturbing force will therefore tend to diminish the velocity of the particles, and cause them to move in a more southerly direction. As the moon rises her disturbing force increases, and changes its direction more towards the south, though very slowly during the first three hours from moonrise; therefore the obtuse angle between the two directions will diminish, pass through a right angle, and become acute, thus causing the velocity of the particles to pass through a minimum, and produce a high water, at some time between moonrise and her passage across the particles' meridian. After this the acute angle between the two directions will continue to diminish until they coincide, the moon's true bearing continuing to change in the same direction, but more rapidly as she approaches the meridian. After coincidence it will cross the direction of the particle's motion, and make a rapidly increasing *acute* angle with it on the opposite side, drawing the particles of water towards it, and causing the direction in which they move to change in such a manner as to make it revolve in the same direction as the true bearing, but much more slowly. When the moon is within one hour of the meridian her true bearing changes very rapidly, so that the angle between the two directions will increase very rapidly from an acute angle, pass through a right angle, and become obtuse at some time *after* the moon has crossed the meridian. At this time the velocity of the particles of water will pass through a maximum and produce a low water. About one hour after the moon has passed the meridian, the moon will bear from the particle S. 57° W., and the rate of the change in the bearing will have been reduced to its mean value, and afterwards the change will be slower and slower; whilst the rate of change in the direction of the particle's motion will con-

tinually increase; therefore after some time the rates at which the two directions are changing will become equal, and the obtuse angle between them no longer increase. The change in the direction of the particle's motion after this being greater than that of the moon's true bearing, the obtuse angle between them will begin to diminish, pass through a right angle, and become acute, giving another minimum velocity to the particles of water, and therefore another high water. From this time until moonset, when the disturbing force ceases, the acute angle between the direction of the force, and that in which the particles of water are moving, though continually diminishing, always remains acute; the velocity of the particles will therefore continually increase until shortly *before* moonset; the velocity of the particles will then reach another maximum and a second low water will happen. During the time the moon is below the horizon, viz., from moonset to moonrise, the action of her disturbing force and its results will be exactly similar to that we have just described as taking place whilst she was above the horizon.

We may therefore expect when the moon is on the equator to find, in a parallel of latitude belt of a free ocean of which the latitude is 10°, either north or south, the surface of the sea rising soon after moonrise, and continuing to rise with the moon until it attains a high water some time *before* the moon reaches the meridian, when it will fall to a low water, which will occur some time *after* the moon has crossed the meridian; it will then rise to a second high water and fall again to a second low water, which will happen some short time *before* moonset; the surface will afterwards begin to rise and attain a third high water at some time *before* the moon's inferior transit, it will then fall to a third low water, which will happen *after* the moon's inferior transit; again rise to a fourth high water, and fall to a fourth low water, which will take place shortly before moonrise. Thus the moon's disturbing force will here produce during a lunar day four tide waves of unequal size and period, the high waters of the two larger waves happening *before* the moon's transits, and those of the two smaller waves *after* the moon's transits respectively.

The effect which the moon's disturbing force will produce on the ocean in a parallel of latitude 45° N. or S. of the equator, must next be considered, the moon still remaining on the equator. Her true bearing at rising in this latitude is east, at

one hour after moonrise	it will	be	S. 80° E.
two hours	„	„	S. 68 E.
three hours	„	„	S. 55 E.

four hours after moonrise it will be			S. 39½ E.
five hours	„	„	S. 21 E.
six hours	„	„	South;

and the moon will have the same number of degrees of bearing between moonset and her upper transit, for each hour before moonset, only writing W. for E.

The moon's horizontal disturbing force will here increase from zero when she rises to its maximum value when she crosses the meridian, after which it will decrease, becoming zero when she sets.

At moonrise the particles of water will be moving in a direction to the *southward* of west, and her disturbing force then commencing from zero to act to the eastward, inclining towards the south gradually as it increases, will make a gradually decreasing obtuse angle with the direction in which the particles of water move; when it becomes a right angle the velocity of the particles will then be at a minimum, and there will be a high water at some time *before* the moon passes the meridian; after passing a right angle the acute angle between the two directions will gradually diminish until they coincide; the moon's true bearing then crossing on the other side of the line of direction in which the particles of water are moving forms a gradually increasing acute angle, which will continue to increase until the rate of change in the line of direction in which the particles move, which is now in the same direction as the true bearing, becomes equal to the rate at which the moon's true bearing changes, when the angle between the two directions will attain its largest size and will afterwards diminish. If we examine the moon's true bearings for the different hours that she is above the horizon, as noted above, we shall quickly perceive that the change in the moon's true bearing is not sufficiently large to enable it to gain by a right angle upon the line of direction in which the particles of water move, between the time when the two directions coincide, and the probable time at which the angular velocity of the line of direction in which the particles move is as large as that of the moon's true bearing; so that the angle between the two directions will remain acute until moonset, when the disturbing force ceases, and therefore the velocity of the particles will increase until a short time before moonset, when there will be a low water. In like manner there will be a second high water after moonset and before her inferior transit, and a second low water some short time before moonrise.

We may therefore, in these latitudes, under the foregoing circumstances, expect to find two equal and similar lunar

waves during a lunar day, the high waters of which will happen *before* the moon's transits respectively, and the low waters shortly before moonrise and moonset respectively.

We conclude from the foregoing description that when the moon is on the equator she will produce, in a free ocean, four equal and similar tide waves during a lunar day, at places immediately under the equator. Proceeding from the equator towards the poles the four waves no longer remain all equal to each other, but only the alternate waves; each of the larger waves, which are equal to each other, being followed by one of the smaller waves, which are also equal to each other, and so on alternately. The difference between the two larger waves and the two smaller waves increases with the latitude of the place until, at a certain distance from the equator, the small waves become so small as to be insensible and leave only the two equal and similar large waves during a lunar day.

Before describing the general effect of the moon's disturbing force, when she is either north or south of the equator, it will be convenient to state and prove two important propositions.

1. The moon's horizontal disturbing force on particles situated in a given parallel of *north* latitude, which have the moon *above* their horizon, and on particles situated in the parallel of south latitude of the same degree, which have the moon *below* their horizon, may for all practical purposes be considered the same, and *vice versa.*

2. When the moon is in a given declination *north*, her horizontal disturbing force upon all particles in a given parallel of latitude *above* whose horizon she is, and that which she would have exerted upon the same particles if she had been *below* their horizon, and in south declination of the same degree, may also be considered equal, and *vice versa.*

Let $NMESmW$ (Fig. 44) be a section of the earth, by a plane passing through the moon, and O the centre of the earth, M the point on the earth's surface which has the moon in its zenith, and m the point which has the moon in its nadir, NOS the axis of the earth, E and W the east and west points. Let P be a particle at the earth's surface in a given parallel of north latitude, p one in the same degree of *south* latitude, but whose longitude differs from that of P by 180°. Draw the meridian $NPSp$ passing through P and p, and join PM and pm by arcs of great circles.

The moon's horizontal disturbing force on P is equal to $M \sin 2PM$, where M is the whole disturbing force pulling the particle relatively towards the moon.

The moon's horizontal disturbing force on p at the same instant is equal to $M \sin 2pm$, but $pm = PM$, and therefore

the moon's horizontal disturbing force on p is equal to that on P.

If the moon, instead of being in north declination and in the zenith of PM and the nadir of m, had been in south declination of the same degree and in the zenith of m, she would have been below the horizon of P, and her horizontal disturbing force on P would have been equal to $M \sin 2PM$, the same as when in her previous position in the zenith of M.

Hence, for all practical purposes, when the moon is off the equator, we may consider that her horizontal disturbing force will during a lunar day produce the same effect on all particles in the same degree of latitude independently of its name.

For the sake of distinction we shall call the waves formed by the moon's disturbing force whilst she is *above* the horizon

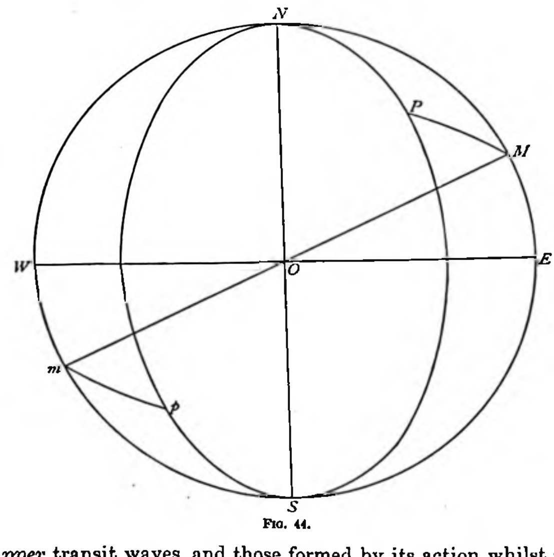


FIG. 44.

upper transit waves, and those formed by its action whilst she is *below* the horizon *lower* transit waves. The foregoing can therefore be stated as follows.

The upper transit waves formed in any parallel of latitude in the northern hemisphere and the lower transit waves formed in the corresponding parallel of latitude in the southern hemi-

sphere are equal to each other, and *vice versa*, other things being the same. This enables us to confine our attention to one hemisphere when the moon is off the equator; we will therefore take the northern hemisphere, with the moon north of the equator. Keeping this in mind, when the moon moves from the equator towards the pole, by a similar process of reasoning to that employed when she was on the equator, we should conclude that at places near the equator there will be four lunar waves during each lunar day, but that the sizes and periods of the two upper transit waves will be respectively larger than the lower transit waves which correspond to them, because the moon will be longer above than below the horizon during a lunar day. At places further north, in latitude 45° suppose, there will probably be only two waves of unequal size and period during a lunar day, the upper transit wave being the larger, the difference in the size and periods of the two waves increasing with the latitude, whilst in the same place the difference will increase with the moon's declination. At places near the pole, when the moon's declination is large, the smaller or inferior transit waves will get so small as to be insensible, leaving only the superior transit waves during a lunar day, which will gradually diminish as the moon's declination increases, and finally disappear; so that in the Arctic circle when the moon's declination is large we may conclude there will be no diurnal wave.

We may therefore expect to find at places near the equator four lunar diurnal waves, at places about 45° of latitude two lunar diurnal waves; that when the moon is on the equator the four lunar diurnal waves will only be equal at places immediately under the line, whilst at places sufficiently far from the equator to have only two lunar diurnal waves, these waves will be equal.

As the moon moves from the equator to the pole the superior transit waves at places on the same side of the equator that she is will be larger than the inferior transit waves and their periods longer, whilst at places on the other side of the equator the inferior transit waves will be larger in size and of longer period than the superior transit waves. At places near the pole on the same side of the equator as the moon, the inferior transit waves will gradually diminish as the moon moves towards the pole, until it disappears—leaving only the superior transit wave, which diminishes as the moon moves towards the pole, until it also disappears, leaving no semi-diurnal wave. At places on the opposite side of the equator from the moon a similar succession of lunar waves will occur during each lunar day, the only difference being that the inferior transit waves

take the place of the superior transit waves in the other hemisphere, and *vice versa*.

Besides the lunar diurnal waves just described, which are due principally to that part of the moon's horizontal disturbing force resolved perpendicular to the meridian, the total action of which during a lunar day may be considered as zero, there is another fluctuation in the surface of the ocean due to the resolved part of the moon's disturbing force parallel to the meridian, which remains uncompensated for during a lunar day, leaving a residue acting towards the equator. This force, besides varying as the cube of the moon's horizontal parallax, is otherwise greatest when the moon is on the equator and least when she is farthest from it for all places in the same latitude; but when other things are the same it is greatest for places in latitude 45°, whilst at the poles and at the equator it has no effect, whatever may be the moon's declination. The particles of water are thus drawn towards the equator from the poles, and meeting at the equator from opposite sides of it cause the surface of the ocean to rise higher there than it otherwise would, whilst at the poles the water will fall below the level it would otherwise have. If this force always remained constant in the same latitude, the surface of the ocean, after attaining a position of equilibrium under its influence and that of the other constant forces, would remain unchanged so far as this force is concerned, but as it varies the surface will rise and fall from the position due to the mean value of this force if it continued constant, standing highest at the equator and lowest at the poles when the moon is on the equator and nearest to the earth, and highest at the poles and lowest at the equator when she is farthest from the earth and also from the equator. At places intermediate between the equator and the poles in the neighbourhood of latitude 45°, the water will be moving from the pole towards the equator when the surface is rising there, and from the equator towards the poles when the surface is falling at the equator, and in these latitudes the surface of the ocean where unaffected by land will probably rise whilst the water is running towards the poles and fall whilst it is running towards the equator; but this probability, which is derived from a consideration of the figure of the earth, will be modified by the varied rates of motion from the variations in the disturbing force.

It is found that, when the particles of water are set vibrating by any force, their motion does not cease immediately upon the force ceasing, but that the particles continue to vibrate in such a manner that when a wave is once formed, it is followed by a series of waves, nearly the same in period but of gradually diminishing size, so that after a time they become insensible:

these are called free waves to distinguish them from the wave formed by the immediate action of the force, which is called a forced wave. Each of the lunar waves formed under the immediate action of the moon's disturbing force, in the manner before described, will therefore be followed by a series of free waves diminishing in size from, but of nearly the same period as, the forced wave from which they are derived; and the whole lunar wave will be a combination of the forced wave and the free waves which follow respectively each of the previous lunar waves formed by the moon's disturbing force.

It is obvious that even under the favourable conditions we have assumed, in order to simplify our description, it would be impossible to form a correct estimate even of the number of the lunar diurnal tide waves at a given place, much less of their periods and ranges, and that careful observations must be made continuously at least through one lunation, before even a rough approximation can be arrived at. In addition, we must consider the effect an island is likely to produce on an oceanic tide wave meeting its shore, dividing it into two waves passing round the island in opposite directions and again meeting each other at different parts of its shore in different phases, at some combining their effects, at others neutralising them; and consider also that every change on the shore has an influence on the wave moving along it, both as regards the size of the wave and the rate at which it moves. Some of these local influences are enormous, for instance, the range of the tide at the head of the Bay of Fundy is about eight times as great as the range at its mouth, whilst it is full two-thirds ebb at the mouth of the bay when it is high water at the head. We are thus led to the conclusion that careful observations on the tides are absolutely necessary if we wish to form an estimate of the state of the tide at any future period.

The sun produces a disturbance on the ocean similar to the moon, but its immense distance from the earth more than counterbalances its enormously greater attraction, so that the sun's disturbing force is much less than that of the moon, and the solar waves though similar to the lunar waves are much smaller.

The oceanic tide wave is a combination of the lunar and solar waves and is therefore more complicated in its movements than either of them taken separately. We may, however, form a good general idea of the oceanic tide wave by considering it composed of two waves, one following the lunar and the other the solar transits, each fluctuating with the changes in the positions and distances of their governing bodies. When the motions of the sun and moon, relatively to the earth, bring them into the same straight line as seen from the earth, or

exactly opposite to each other with the earth between them, their disturbances acting together produce the large waves called spring tides; but when the positions are such that the straight lines drawn from the earth to the sun and moon respectively form a right angle with each other, the low water of the solar wave happening about the high water of the lunar wave and *vice versa*, the small waves are produced called neap tides.

During a lunar month the moon's disturbing force passes through its monthly cycle of changes whilst the solar disturbing force alters but little, and we can therefore consider it to remain constantly the same during a lunar month at the value it has at the middle of the month. This enables the effect of the solar wave to be approximately eliminated from the observed fluctuations of the combined wave, and the solar wave determined with sufficient accuracy for the general purposes of navigation. The fluctuations which remain can be considered as those of the lunar wave and be equated to expressions involving the moon's elements, approximating, as near as simple expressions can, to the complicated formulæ which, if we could determine them, would exactly represent the semi-tidal movement. From these equations the values of quantities, which as far as a first approximation goes may be considered constant, are found, from which corrections in terms of the lunar elements can be tabulated. These when applied to the time and height of the mean lunar wave will give the approximate time of high water and the range of the lunar tide wave. The corrections due to the sun's wave can then be applied, and the time of high water and the range of the tide determined sufficiently near for practical purposes.

Before describing the mode of observing tides, it will be convenient to say a few words regarding the level of the zero to which the tides should be referred in order that the soundings taken from time to time at different places and when the surface of the sea is at different levels, may all be reduced to the same level before they are inserted on the chart.

The mean tide level is exactly half way between the averages of the heights of the high waters and the low waters taken respectively through thirteen successive lunations. In places where there are two regular diurnal tides, one fourth the sum of the headings of any two high waters immediately following each other and their corresponding low waters taken during fine moderate weather will generally give the reading of the mean tide level within a few inches, so that its situation can generally be quickly and easily determined with sufficient accuracy for ordinary purposes. It is useful to compare tides at

different places with each other in order that all the tide boards and tide gauges throughout the survey may have their zeros so determined that, when one of them shows zero for the low water of a given tide, all the others will show zero for the low water of the same tide. The zero usually adopted and the one used in the Admiralty charts is the low water level of an average spring tide. This is indefinite and not recoverable, and therefore, when the zero is determined, the reading of the half tide line corresponding to it should be inserted on each sheet.

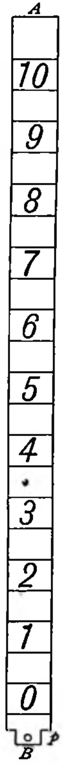


FIG. 45.

The tide board AB (Fig. 45) is made of soft wood, about 2 inches thick, 6 inches broad, and 12 feet long, painted white; across each face narrow straight black lines, perpendicular to the length of the board, are drawn, exactly 6 inches distant from each other, measured from the lower edge of the board; between the first and second lines, counting from the bottom, the figure 0 is painted black in the middle of the face and touching the two black lines between which it stands, between the third and fourth lines from the bottom the figure 1 is painted in a similar manner, between the fifth and sixth lines from the bottom the figure 2 is painted, and so on until the faces of the board are filled from bottom to top. Each figure is taken as the reading in feet of the straight line on which it stands, and the straight line running across the top of each figure is to be read as 6 inches more than the number of feet the figure denotes: thus, the straight line upon which the numeral 6 stands is read 6 feet, and the straight line running across its top 6 feet 6 inches, and so on.

When a wharf or other stable structure can be found with sufficient water at low water spring tides, the board can be secured to it with its faces vertical, otherwise a frame CDE (Fig. 46) made of rough pieces of hard wood must be made capable of holding the tide board securely in a vertical position; the frame CDE is placed on the bottom at low water and well ballasted so as to keep it firmly fixed there, remembering always that a sandy bottom will not do. F is a socket cut in the side CD to receive the lower end of the tide board, the mortice at B (Fig. 45) fitting loosely into it and secured with a pin passing through the hole p, so as to allow AB sufficient play to enable it to be fixed in a vertical position by stays originating from the corners C, D, and E respectively.

In places where the range of the tide exceeds the length of the tide board another board must be placed higher up on the beach in a similar manner, so that the surface of the water, when rising, may reach its zero before the top of the first board is covered, in order that the two boards may be read simultaneously, and the two boards should always be read together as long as possible, both in the

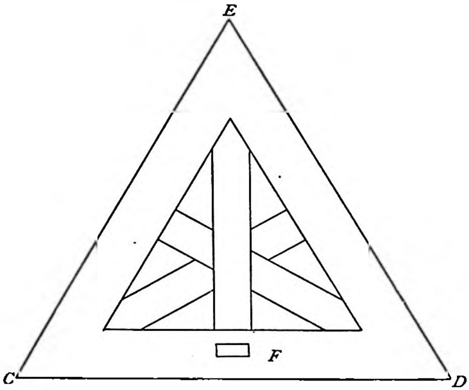


FIG. 46.

rising and falling tide, and the reading noted in the tide book with the time; sometimes more than two tide boards will be required, when a third board must be placed on the beach in a similar manner and the same process as to reading them together observed, and so on. On one occasion at the head of the Bay of Fundy the range of the tide was so large that we used *five* tide boards in succession on the beach.

On commencing the survey of any place a tide board should be erected as soon as possible, and the tide observed every half hour, day and night, throughout two or three lunar days, until the general behaviour of the tide, the approximate position of the mean tide line, and a fair zero for the reduction of the soundings have been determined; after which, if the tides are regular, observations during the middle of a tide may be omitted, except when required for reducing soundings, and the observers' attention devoted to ascertaining carefully the times and heights of the high and low waters as they follow in succession for at least six weeks.

If the zero of the survey is settled and known for any place,

the correction to the reading of the tide board at any other place, necessary to make its zero correspond to that of the survey, is found by observing at *both* places, on a calm fine day, the heights of the high and low waters of two consecutive tide waves, supposing there are *two* diurnal waves; add together the four heights observed at each place respectively and divide the two results by 4, this will give the tide pole reading at each place of its mean tide line corresponding to the same tide waves. Subtract the sum of the two low water readings at each place from the sum of the two high water readings at the same place. The result will be the sum of the two ranges, observed at each place, and the readings of the mean tide lines at each place should be respectively proportional to them. For example—

On the 12th March, 1847, the tide gauge at St. Johns, New Brunswick, read as follows—

Low water, - -	3 ft. 0 in.	Consecutive tides.
High „ - - -	23 ft. 0 in.	
Low „ - - -	2 ft. 3 in.	
High „ - - -	23 ft. 3 in.	
Sum, - - -	51 ft. 6 in.	
Mean tide line, -	12 ft. $10\frac{1}{2}$ in.	

Sum of low water readings, - -	5 ft. 3 in.
„ high „ „ - - -	46 ft. 3 in.
Sum of the two ranges, - -	41 ft. 0 in.

At Campobello, New Brunswick, the same tides were observed with a gauge of which the zero had been determined. The gauge readings were as follows—

Low water, -	3 ft. 8 in.	Same consecutive tides as above observed at St. Johns.
High „ - -	21 ft. 8 in.	
Low „ - -	3 ft. 3 in.	
High „ - -	21 ft. 10 in.	
Sum, - -	50 ft. 5 in.	
Mean tide line,	12 ft. $7\frac{1}{4}$ in.	

Sum of low water readings, - -	6 ft. 11 in.
„ high „ „ - -	43 ft. 6 in.
Sum of ranges, - - - -	36 ft. 7 in.

But 36 ft. 7 in. : 41 ft. 0 in. :: 12 ft. $7\frac{1}{4}$ in. : Reading mean tide line at St. Johns = 14 ft. 1 in.

∴ Correct reading mean tide line St. Johns,		14 ft. 1 in.
Tide gauge reading,	- - -	12 ft. $10\frac{1}{2}$ in.
Correction to St. Johns tide gauge,		+ 1 ft. $2\frac{1}{2}$ in.

A good practical zero for reducing the soundings, placing the tide boards, and correcting their readings should be determined as soon as possible after commencing a survey. A good one for Atlantic tides may be found as follows:—The first time the moon passes the meridian near noon or midnight, after the tide board has been erected, take the readings of two consecutive high waters and also of the low waters which immediately follow the high waters respectively; subtract the sum of the two low water readings from the sum of the two high water readings, divide the difference by four and call the result S; take the difference between the two high water readings and that of the two low water readings, add the results and call the sum D. Then, if p be the moon's horizontal parallax expressed in minutes and δ her declination when it is not less than 6°, the reading of the half tide line corresponding to a good practical zero will be $\frac{244-3p}{61}\left\{S+\frac{D}{10\sin\delta}\right\}$.

Example. At Halifax, Nova Scotia, in October, 1858, the reading of the tide gauge was—

7th.	8^h 0^m	P.M.—High water,	- -	7 ft. 0 in.
8th.	2 10	A.M.—Low water,	- -	2 ft. 3 in.
„	8 45	„ —High water,	- -	7 ft. 7 in.
„	3 30	P.M.—Low water,	- -	2 ft. 3 in.
				19 ft. 1 in.
		Reading mean tide line,	- -	4 ft. $9\frac{1}{4}$ in.
		Sum of high water readings,	- -	14 ft. 7 in.
		„ low „ „	- - -	4 ft. 6 in.
				10 ft. 1 in.
				S = 2 ft. $6\frac{1}{4}$ in.

Difference of high water readings,	-	0 ft. 7 in.
„ low „ „	- -	0 ft. 0 in.
	$\therefore D=$	0 ft. 7 in.
		$=0{\cdot}58$ ft.

The moon's horizontal parallax being 57′ and her declination 11° S. a good reading of half tide line

$$=\frac{73}{61}\{2{\cdot}521+0{\cdot}058\times\text{cosec }11^\circ\}\text{ft.}=3\text{ ft. }5\text{ in.}$$

To calculate the above we have

log 0·058, - - -	$\bar{2}{\cdot}763$	log 2·825, - - -	0·451
log cosec 11°, - -	0·719	log 73, - - -	1·863
		AC log 61, - -	$\bar{2}{\cdot}215$
log 0·304, - - -	$\bar{1}{\cdot}482$		
2·521		log 3·41, - - -	0·529
		(or 3 ft. 5 in.)	
2·825			

Gauge reading of mean tide line, - -	4 ft. $9\frac{1}{4}$ in.
A good zero gives - - - - -	3 ft. 5 in.
Correction to tide gauge reading, -	1 ft. 4 in.

In the selection of a zero for sounding the fractions of an inch are not important and have therefore been neglected.

At Yarmouth, Nova Scotia, 20th June, 1853, the following consecutive observations were made. The moon passed the meridian at $11^h\ 57^m$ P.M., when her horizontal parallax was 61′ and her declination 64° S.

20th June.—High water, $9^h\ 56^m$ A.M., -	14 ft. 7 in.
Low water, - - -	2 ft. 2 in.
High water, $10^h\ 18^m$ P.M.,	16 ft. 9 in.
Low water, - - -	0 ft. 9 in.
	34 ft. 3 in.
Tide board reading mean tide, -	8 ft. $6\frac{3}{4}$ in.

Reading for a good zero $=\{7{\cdot}1+0{\cdot}358\text{ cosec }24^\circ\}$ ft.
$=7{\cdot}1+0{\cdot}88=8$ ft.

For	log 0·358, - - - - - -	$\bar{1}$·554
	log cosec 24°, - - - - -	0·391
	log 0·88, - - - - - -	$\bar{1}$·945

Therefore correction to tide board = − 7 inches.

Here the sun is in great north declination, and the moon passes the meridian with a large south declination near midnight, so that the disturbing bodies are very nearly opposite to each other, and the second term allows to a very great extent for the effect of both their diurnal inequalities.

If a good self-registering tide-gauge can be procured, it should be erected in a good sheltered position to which the tide wave has as free an access as possible; it should be kept constantly at work as a standard to which all the other tide observations should be referred.

Tide boards should be placed in pairs at convenient stations on the coast, one of each pair in a sheltered spot where it is well protected from the sea, and the other in a more exposed place, selected on account of the tide wave having a perfectly free access to it; when the weather is fine and smooth both boards must be observed simultaneously throughout several tides, when the comparison will show whether there is any change in the time and height at which the tide wave reaches the two boards, and when this local difference—if it exist—is determined, the observations need only be made on the sheltered board.

When the "Columbia" was employed under the late Admiral W. F. W. Owen, surveying the Bay of Fundy, Messrs. Urquhart and Cowley, her engineer officers, constructed in 1844 two very good self-registering tide-gauges, on a plan which I suggested, and which they executed with such ingenuity and mechanical skill that the Admiralty gave a medal to the former and a gratuity to the latter, besides directing them to make a similar gauge for the naval yard at Halifax, which they did in 1845, when it was set up there and registered the tides accurately for many years.

When a self-registering tide-gauge is out of the question, one of the tide boards should be taken as a standard, and at this observations should be continually made, or at least consecutive observations for the times and heights of high and low water respectively should be made from about 1st June to 12th July, from 1st September to 12th October, from 1st December to 11th January, and from 1st March to 11th April.

The wind, weather, state of the sea, barometer, and thermometer should be observed and noted in the tide book, as well as

the average force and direction of the wind during each tide, noting particularly if any sudden change takes place in the wind, as well as estimating and noting the relation between the direction of the wind and the height of the surface of the sea.

The tide register should be copied from the tide book, and in it the following elements of the moon and sun should be inserted:—Moon's meridian passage, horizontal parallax and declination corresponding to the time of her meridian passage, sun's declination, semidiameter, and the equation of time at apparent noon.

As soon as the tide zero at any place has been settled, horizontal lines should be cut upon the wharf, or upon a rock, or other *fixed* object, at some known vertical distance above the zero, in order that should any accident remove the tide board from its place, the zero can easily be recovered; besides the reading of the tide board should always be taken and noted when the top of any conspicuous rock near it covers on the rising tide, and uncovers when it falls; a sketch of the rock shortly before it covers, as seen from the place of observation, should be made in the tide book.

A daily comparison should be made between the tide watch and the standard chronometer when practicable, the watch set to mean time, or its error in mean time noted in the tide book. When this cannot be done the error of the watch can easily be found by equal altitudes of the sun taken near noon with a sextant and artificial horizon; the observation is very simple, and quickly made in the following manner:—About ten minutes before apparent noon take the sun's altitude, noting the time by the watch; about two minutes after take a second altitude of the sun; keep the instrument firmly clamped at this altitude, noting the time as before; wait and watch until the sun has passed the meridian and fallen to the altitude clamped on the sextant, when again note the time; reset the instrument to the first altitude, and when the sun has fallen to it again note the time by the watch; half the sum of the times corresponding to the same altitude would, if the sun did not move in declination, be the watch time of apparent noon; but in consequence of this motion a small correction must be applied to the half sum to give the exact time of apparent noon; generally it is too small to make any practical difference, as will be seen from the following table, where the correction is noted to the nearest second.

TABLE OF CORRECTIONS TO BE APPLIED TO THE HALF SUM OF THE TIMES.

Latitude of Place.	June and December.	May – July +	April – August +	March – Sept. +	Feb. – Oct. +	Jan. – Nov. +
60° N.	0^s	9^s	20^s	26^s	27^s	16^s
45 N.	0	4	10	15	16	10
30 N.	0	2	5	8	10	7

In the table north latitude has been taken, but for south latitude we have only to change the sign of the correction. Thus, when in south latitude the correction in January, February, March, April, and May must be added instead of subtracted; and in July, August, September, October, and November it must be subtracted instead of added.

Take the following example: 9th October 1855, in latitude 44° 30′ N., the following observation was taken, the equation of time being $12^m\ 40^s$ to be subtracted from apparent time.

2 ☉'s Altitude.	Time A.M.	Time P.M.	Sum of Times.
° ′	h m s	h m s	h m s
77 58	11 49 20	12 6 35	23 55 55
78 0	11 51 17	12 4 37	23 55 54

Mean, - - - - -	$23^h\ 55^m\ 54^s{\cdot}5$
Half sum, - - - - -	11 57 57
Correction from Table I., -	+ 0 16
Watch time of apparent noon,	$11^h\ 58^m\ 13^s$
Equation of time, - -	+12 40
Watch fast of mean time, -	$10^m\ 53^s$

To find the time of high or low water from the observations recorded in the time book, take a sheet of blue lined paper, the equidistant parallel blue lines to represent the *times* of the observations to be used; write these times across the lines which are to represent them, the left hand line denoting the earliest time, and so on in succession in proceeding from left to right; across the equidistant and parallel blue lines draw a straight line perpendicular to them, and let it represent the foot line on the tide pole corresponding to the first observation

to be used if high water is to be found, and that which is next greater if a low water is to be determined. In the former case subtract the number of feet the straight line denotes from each observation in succession, expressing the differences in feet and inches; in the latter case subtract the readings of the tide pole from the number of feet the straight line denotes, and express the differences in feet and inches. Using an inch scale subdivided to $\frac{1}{12}$ths, and a pair of dividers, take 1 inch on the scale to represent a foot of the tide board, and with the dividers take from the scale the distance corresponding to the first difference, and lay it off on its blue time line from the point where the straight foot line intersects it, above the line for high water and below it for a low water; the point thus defined will express the height of the tide at the time denoted

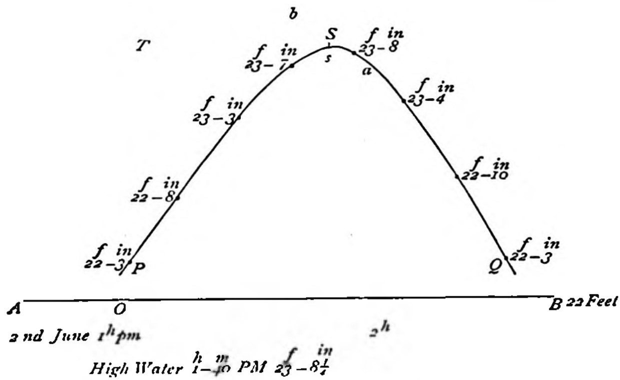


FIG. 47.

by the blue line on which it is. Proceeding in the same manner with the succeeding observations, a series of points will be found showing the height of the tide at the times denoted by the blue line on which they are; through these points with a free hand draw a fine line of *continuous* curvature; in case of high water the highest point of the curve will denote it, and for a low water the lowest point of the curve will give the time and height of the low water.

The following examples with the figures will show how this is done.

At the entrance to the Magaguadavic River, New Brunswick, on the 2nd June 1847, the following observations were made.

At 0^h 45^m P.M. reading of tide board			22 ft.	3 in.,
1 0	”	”	22 ft.	8 in.,
1 15	”	”	23 ft.	3 in.,
1 30	”	”	23 ft.	7 in.,
1 45	”	”	23 ft.	8 in.,
2 0	”	”	23 ft.	4 in.,
2 15	”	”	22 ft.	10 in.,
2 30	”	”	22 ft.	3 in.

Between 1^h 30^m and 1^h 45^m P.M. the highest point 23 ft. $8\frac{1}{4}$ in. was reached.

Draw the straight line AB (Fig. 47) perpendicular to the equidistant parallel blue straight lines, and take it to represent the 22 feet line on the tide board, also take the blue straight line OF to represent 0^h 45^m P.M., the next blue straight line to the right of OT to represent 1^h P.M., and so on in succession. Subtracting 22 feet from the tide-board reading at 0^h 45^m P.M. we have 3 inches. Therefore on OT lay off $OP = \frac{3}{12}$ inch; the point P will denote that at 0^h 45^m P.M. the height of the tide was 22 feet 3 inches; in a similar manner on the next blue straight line to the right of OT lay off above AB $\frac{8}{12}$ inch, the point thus determined will represent that the height of the tide at 1 P.M. was 22 ft. 8 in.; proceeding in this way we obtain the points expressing the height of the tide corresponding to each observation. Draw the continuous curve PSQ through or as near as possible to all the points in succession; the highest point S will show the time of high water, which is determined as follows. From the half inch scale divided diagonally to tenths and hundredths, supposing that half an inch is a *little longer* than the distance between two blue lines, take off with a pair of dividers half an inch, place one leg of the compasses at the point a on the blue time line 1^h 45^m, and with the extremity of the other leg describe a circular arc cutting the blue time line 1^h 30^m, in the point b; join ab by a straight line. The point a is selected so that the straight line ab shall pass near to the point S, through which draw the straight line Ss parallel to the blue lines cutting ab in s, measure the shortest of the two lengths as or bs, in this case as is the shortest and measures on the half-inch scale 0·33; but $15^m \times \frac{as}{ab} = 5^m$ is the number of minutes high water happens before 1^h 45^m P.M.; therefore 1^h 40^m P.M. is the time of high water. It can also be done mechanically by subdividing the straight edge of a piece of blue lined paper which lies between *two* blue lines into fifteen equal parts, and placing the edge of the paper diagonally between the two blue time lines con-

taining the time of high or low water to be found, so that the extreme blue lines of the subdivided edge coincide with the two blue time lines, and then slipping the paper up and down until it passes through the point defining the high or low water to be determined; when the number of subdivisions this point is from either of the time lines between which it lies can be counted, and will be the number of minutes to be added to the time of the left hand line, or subtracted from that of the right hand line as the case may be.

On the 3rd June the following low water projected in a

Scale—One inch to a foot.

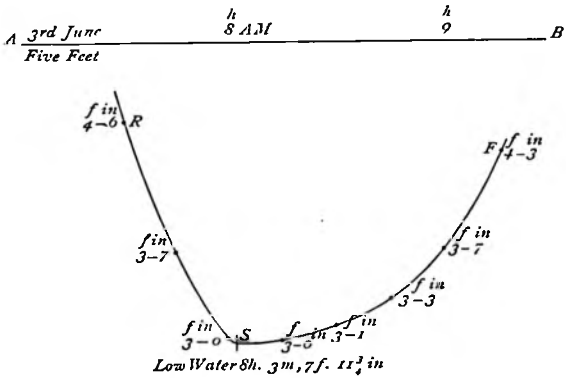


FIG. 48.

similar way in Fig. 50 was observed. The lowest point to which this tide fell was 2 ft. 11¾ in. shortly after 8h A.M.

At 7h	30m A.M. reading of tide board was			4 ft. 6 in.,
7	45	,,	,,	3 ft. 7 in.,
8	0	,,	,,	3 ft. 0 in.,
8	15	,,	,,	3 ft. 0 in.,
8	30	,,	,,	3 ft. 1 in.,
8	45	,,	,,	3 ft. 3 in.,
9	0	,,	,,	3 ft. 7 in.,
9	15	,,	,,	4 ft. 3 in.

We will now give a few examples of the tides observed on the S.E. coast of Nova Scotia, where the range of the tide is very much smaller than the foregoing.

At Prospect, Nova Scotia, the following observations for high and low water were made on the 31st May, 1863.

Time.		Height.	
Hours.	Minutes.	Feet.	Inches.
11	30 A.M.	2	1
11	45	1	10
Noon.		1	10
0	15 P.M.	1	10
0	30	1	10
0	45	1	10
1	0	2	1

Time.		Height.	
Hours.	Minutes.	Feet.	Inches.
5	0 P.M.	6	4
5	30	6	5
6	0	6	7
6	15	6	6
6	30	6	8
6	45	6	7
7	0	6	7
7	30	6	4

Scale—One inch to a foot.

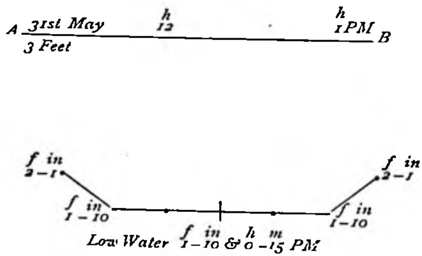


FIG. 49.

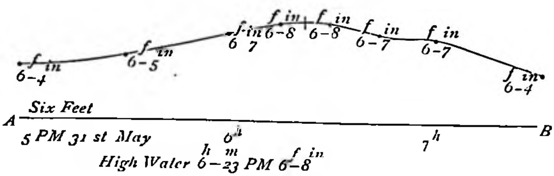


FIG. 50.

Fig. 49 is the projection of the low water of the 31st May, and Fig. 50 that of the high water of the same day.

7TH JUNE, 1863, PROSPECT, NOVA SCOTIA.

Time.		Height.	
Hours.	Minutes.	Feet.	Inches.
11	30 A.M.	5	7
11	45	5	9
Noon.		5	11
0	15 P.M.	6	0
0	30	5	8
0	45	5	11
1	0	5	9
1	15	5	6

Time.		Height.	
Hours.	Minutes.	Feet.	Inches.
5	45 P.M.	2	0
6	0	1	9
6	15	1	7
6	30	1	7
6	45	1	7
7	0	1	7
7	15	1	9
7	30	1	9
7	45	2	0

Scale—One inch to a foot.

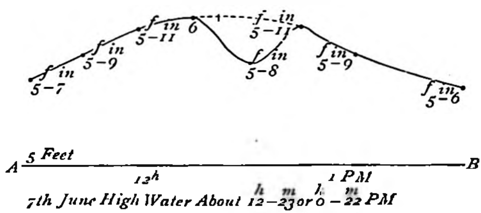


Fig. 51

Scale—One inch to a foot.

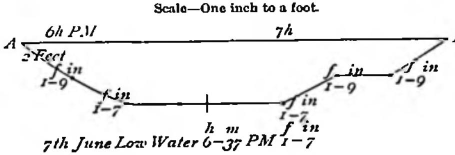


Fig. 52

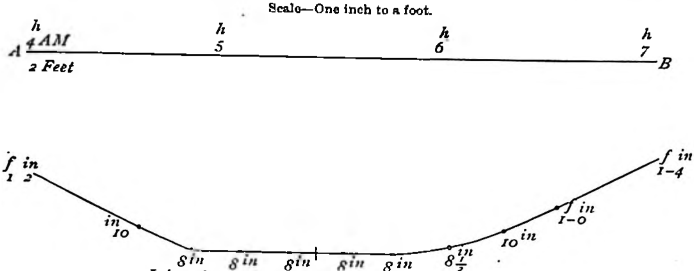
The time of high water on 7th June (Fig. 51) was rendered uncertain to some extent on account of the low water of one of the small waves which sometimes visit the coast of Nova Scotia happening about the time of high water. The dotted line shows the probable curve the tide wave would otherwise have described, and the probable time of high water was or would have been 0^h 22^m P.M.

6TH JULY, PROSPECT, NOVA SCOTIA.

Time.		Height.	
Hours.	Minutes.	Feet.	Inches.
4	0 A.M.	1	2
4	30	0	10
4	45	0	8
5	0	0	8
5	15	0	8
5	30	0	8
5	45	0	8
6	0	0	8½
6	15	0	10
6	30	1	0
7	0	1	4

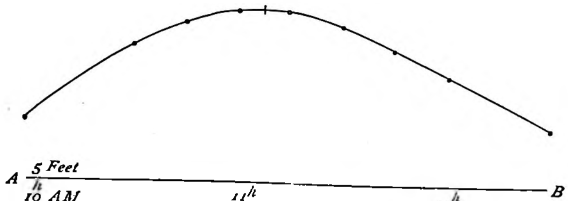
Time.		Height.	
Hours.	Minutes.	Feet.	Inches.
10	0 A.M.	5	5
10	30	5	11
10	45	6	1
11	0	6	2
11	15	6	2
11	30	6	1
11	45	5	11
Noon.		5	9
0	30 P.M.	5	5

Scale—One inch to a foot.



July 6th Low Water 5h-22m AM of-8in

FIG. 53.



6th July High Water 11h-7m AM 6f-2in

FIG. 54.

26TH SEPTEMBER, PROSPECT, NOVA SCOTIA.

Time.		Height.	
Hours.	Minutes.	Feet.	Inches.
5	0 A.M.	5	9
5	30	5	11
5	45	6	0
6	0	6	0
6	15	6	0
6	30	5	11
7	0	5	9

Time.		Height.	
Hours.	Minutes.	Feet.	Inches.
11	30 A.M.	1	4
11	45	1	2
Noon.		0	11
0	15 P.M.	0	11
0	30	0	11
0	45	1	2
1	0	1	4

Scale—One inch to a foot.

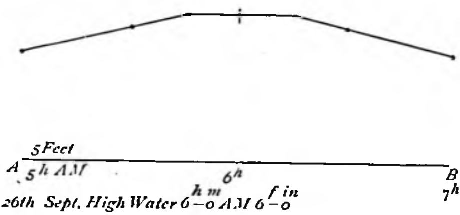


FIG. 55.

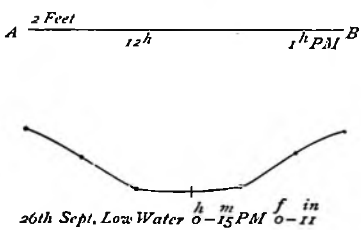


FIG. 56.

At times the S.E. coast of Nova Scotia is visited by a series of small waves, the periods of which vary from five to twenty-five minutes, and their ranges rarely exceed four inches. When

these waves arrive about the time of high or low water they show themselves distinctly and cannot fail to be noticed. The low water of one of these arrived about high water on the 7th June, 1863, and is shown very distinctly in Fig. 51. The irregularities in Figs. 50 and 52 may arise from a similar cause.

When the wind blows on shore it causes the surface of the sea to rise above its ordinary level, and when it blows off shore to fall below the level it would otherwise have had. This effect depends very much on the shape of the shore as well as the strength of the wind, and sometimes amounts to a foot or more.

If the wind changes suddenly near the time of high or low water from blowing directly on shore to blowing off shore or *vice versa*, the change in the direction of the wind will produce a considerable change in the height of the surface of the sea, which, combining with the small tidal change which then occurs, will cause a large alteration in the time of high or low water from that which would have resulted had the weather been calm, and the motion of the surface been due to the tide alone; and the smaller the range of the tide the greater the error from this cause will be.

We shall therefore expect irregularities in the times and heights of high and low water given by observation, and not be surprised with those in the following observations extracted from the tide register of M'Grath's Cove, on the S.E. coast of Nova Scotia, which were made during June and July, 1864, and are inserted in Table I.; the letter *U* in the column containing the time of the moon's passage across the meridian denotes that it is her upper transit, and the letter *L* that it is her lower transit. The moon's horizontal parallax and declination correspond to the time of her upper transit; the former is in minutes to the nearest tenth, and the latter in degrees also to the nearest tenth, which is sufficiently near for our purpose.

Table I.

EXTRACT FROM TIDE REGISTER OF M'GRATH'S COVE, BLIND BAY, NOVA SCOTIA, IN MAY, JUNE, AND JULY, 1864.

Date.	High Water.		Low Water.		Moon's			Luni-tidal Interval
	Time.	Height.	Time.	Height.	Meridian Passage.	Horiz. Parallax	Declination.	
	h. m.	h. m.	h. m.	h. m.	h. m.	'	°	h. m.
May 31, A.M.	...	...	...	...	8 38 U.	58·4	10·6 N.	7 47
P.M.	4 25	6 5	...	...	9 4 L.	...	...	8 13
June 1, A.M.	5 17	5 9	11 17	1 8	9 30 U.	57·9	14·5	7 52
P.M.	5 22	6 5	...	...	9 57 L.	...	...	8 10
2, A.M.	6 7	5 11	0 9	1 1	10 23 U.	57·4	17·5	7 45
P.M.	6 8	6 7	0 10	1 7	10 50 L.	...	...	8 2
3, A.M.	6 52	5 11	0 45	0 8	11 16 U.	56·9	19·5	7 36
P.M.	6 52	6 6	1 7	1 6	11 44 L.	...	...	7 56
4, A.M.	7 40	5 11	2 9	0 9	...	56·3	20·3	7 37
P.M.	7 47	6 3	1 47	1 8	0 10 U.	...	...	7 57
5, A.M.	8 27	5 7	2 27	0 4	0 36 L.	...	...	...
P.M.	8 34	6 3	2 34	1 7	1 2 U.	55·7	20·0	7 32
6, A.M.	9 14	5 11	3 4	0 6	1 27 L.	...	...	7 47
P.M.	9 6	6 3	3 6	2 2	1 52 U.	55·2	18·6	7 14
7, A.M.	9 52	5 9	3 58	1 0	2 16 L.	...	...	7 36
P.M.	10 1	5 8	3 53	2 0	2 40 U.	54·8	16·3	7 21
8, A.M.	10 32	5 6	4 24	0 8	3 3 L.	...	...	7 29
P.M.	10 40	5 7	4 47	2 2	3 26 U.	54·4	13·3	7 14
9, A.M.	11 26	5 9	5 11	1 2	3 48 L.	...	...	7 38
P.M.	11 0	6 1	5 11	2 9	4 10 U.	54·2	9·8	6 50
10, A.M.	12 1	6 2	5 38	1 9	4 32 L.	...	...	7 29
P.M.	11 53	5 3	6 8	2 9	4 53 U.	54·2	5·9	7 0
11, A.M.	...	...	6 31	1 9	5 14 L.	...	...	7 24
P.M.	0 38	5 5	7 16	2 2	5 35 U.	54·3	1·7 N.	7 12
12, A.M.	0 47	4 9	7 17	1 8	5 56 L.	...	...	7 36
P.M.	1 32	5 1	8 17	2 0	6 17 U.	54·6	2·5 S.	7 9
13, A.M.	1 26	4 9	8 5	1 11	6 38 L.	...	...	7 24
P.M.	2 2	5 3	9 9	1 10	7 0 U.	55·1	6·6	7 33
14, A.M.	2 33	4 10	8 57	1 11	7 22 L.	...	...	8 3
P.M.	3 25	5 6	10 8	1 10	7 45 U.	55·8	10·8	8 3
15, A.M.	3 48	5 0	9 55	2 3	8 9 L.	...	...	8 1
P.M.	4 10	5 9	10 48	1 9	8 33 U.	56·5	14·4	8 16
16, A.M.	4 49	5 1	10 56	1 11	8 58 L.	...	...	7 58
P.M.	4 56	6 0	11 30	1 6	9 24 U.	57·4	17·4	8 10
17, A.M.	5 34	5 4	11 37	1 10	9 51 L.	...	...	7 46
P.M.	5 37	6 1	...	...	10 18 U.	58·2	19·4	7 50
18, A.M.	6 8	5 8	0 16	0 10	10 47 L.	...	...	7 21
P.M.	6 8	6 1	0 23	1 9	11 16 U.	59·0	20·3	7 53
19, A.M.	7 9	5 7	1 24	0 6	11 45 L.	...	...	7 32
P.M.	7 17	6 6	1 17	1 5	...	...	...	...
20, A.M.	8 3	6 2	2 3	0 3	0 15 U.	59·6	19·8	7 48
P.M.	8 3	6 10	2 10	1 7	0 44 L.	...	...	7 19
21, A.M.	8 49	6 3	2 56	0 1	1 14 U.	60·0	17·9	7 35
P.M.	8 49	6 8	2 56	1 7	1 43 L.	...	...	7 6

TABLE I.—*Continued.*

Date.	High Water.		Low Water.		Moon's			Luni-tidal Interval
	Time.	Height.	Time.	Height.	Meridian Passage.	Horiz. Parallax	Declination.	
	h. m.	ft. in.	h. m.	h. m.	h. m.	'	°	h. m.
June 22, A.M.	9 42	6 8	3 35	0 9	2 12 U.	60·2	14·7	7 30
P.M.	9 42	6 8	3 50	1 7	2 40 L.	...	...	7 2
23, A.M.	10 28	6 8	4 6	0 6	3 8 U.	60·1	10·6	7 20
P.M.	10 21	6 10	4 45	1 11	3 35 L.	...	...	6 46
24, A.M.	11 7	6 8	5 22	0 10	4 2 U.	59·9	5·7 S.	7 5
P.M.	11 52	6 4	5 14	1 9	4 28 L.	...	...	7 24
25, A.M.	...	...	6 1	1 1	4 54 U.	59·5	0·6 S.	7 14
P.M.	0 8	6 8	6 46	1 9	5 20 L.	...	...	7 13
26, A.M.	0 33	6 2	7 18	1 5	5 45 U.	59·0	4·4 N.	7 18
P.M.	1 3	6 8	7 55	1 11	6 11 L.	...	...	7 16
27, A.M.	1 27	6 4	7 57	2 3	6 36 U.	58·4	9·1	7 29
P.M.	2 5	6 11	8 57	1 8	7 2 L.	...	...	7 27
28, A.M.	2 29	5 11	8 59	1 10	7 27 U.	57·8	13·1	7 19
P.M.	2 46	6 4	9 52	1 7	7 52 L.	...	...	7 32
29, A.M.	3 24	5 5	10 1	2 0	8 18 U.	57·3	16·5	7 43
P.M.	4 1	6 4	11 1	1 4	8 44 L.	...	...	8 4
30, A.M.	4 48	5 4	10 56	2 0	9 11 U.	56·7	18·9	7 52
P.M.	5 3	6 4	11 45	1 1	9 37 L.	...	...	8 15
July 1, A.M.	5 52	5 6	...	...	10 38 U.	56·2	20·1	7 54
P.M.	5 57	6 3	0 0	2 1	10 29 L.	...	...	8 16
2, A.M.	6 45	5 8	0 22	1 1	10 55 U.	55·7	20·2	7 50
P.M.	6 45	6 5	0 52	2 2	11 21 L.	...	...	8 1
3, A.M.	7 22	6 0	1 22	1 2	11 46 U.	55·2	19·3	7 43
P.M.	7 29	6 7	1 30	2 5	...	...	...	...
4, A.M.	8 10	6 5	2 14	1 4	0 11 L.	...	...	7 59
P.M.	7 59	6 8	2 9	2 7	0 35 U.	54·8	17·3	7 24
5, A.M.	8 50	6 5	2 50	1 2	0 59 L.	...	...	7 51
P.M.	8 43	6 5	2 45	2 6	1 22 U.	54·5	14·6	7 21
6, A.M.	9 28	6 2	3 28	1 5	1 43 L.	...	...	7 45
P.M.	9 28	6 3	3 35	2 6	2 7 U.	54·2	11·3	7 21
7, A.M.	10 12	6 2	3 50	1 3	2 28 L.	...	...	7 44
P.M.	10 12	6 1	4 26	2 8	2 50 U.	54·1	7·5	7 22
8, A.M.	10 45	6 2	4 30	1 7	3 11 L.	...	...	7 34
P.M.	10 45	5 10	5 0	2 6	3 32 U.	54·1	3·5 N.	7 13
9, A.M.	11 22	5 9	5 22	1 6	3 53 L.	...	...	7 29
P.M.	11 52	5 6	5 45	2 5	4 13 U.	54·2	0·7 S.	7 39
10, A.M.	...	...	6 7	1 6	4 34 L.	...	...	7 26
P.M.	0 0	5 10	6 22	2 4	4 55 U.	54·5	4·9	8 8
11, A.M.	1 3	6 2	6 24	1 8	5 17 L.	...	...	7 13
P.M.	0 30	5 8	7 39	2 3	5 39 U.	55·0	9·0	7 44
12, A.M.	1 23	5 2	7 31	2 3	6 1 L.	...	...	7 37
P.M.	1 38	5 8	8 16	2 3	6 24 U.	55·7	12·7	7 44
13, A.M.	2 8	6 2	8 23	2 2	6 48 L.	...	...	7 28
P.M.	2 16	5 8	9 8	1 10	7 12 U.	56·5	15·9	7 34
14, A.M.	2 46	4 11	9 46	2 3	7 38 L.	...	...	7 38
P.M.	3 16	5 9	10 8	1 6	8 4 U.	57·4	18·5	8 12
15, A.M.	4 16	4 10	10 16	2 1	8 32 L.	...	...	7 44
P.M.	4 16	5 10	11 9	1 4	9 0 U.	58·4	19·9	8 8
16, A.M.	5 8	5 0	11 31	2 0	...	...	...	...

We see, upon looking down the foregoing table, that two well-defined tide waves arrive on this coast during a lunar day, following each other regularly. Taking the mean of the luni-tidal interval noted in the ninth column, from the 31st May to 30th June inclusive, or throughout one lunation, to the nearest minute, we obtain $7^h\ 35^m$; this is usually called the mean establishment of the port. The largest luni-tidal interval during this lunation was that after the upper transit of the 15th June, when it was $8^h\ 16^m$, and the least, $6^h\ 46^m$, followed the P.M. lower transit of the 23rd June; from which we may infer that if $7^h\ 35^m$ be added to the time of the moon's upper or lower transit, we shall obtain the time of the high water following that transit well within an hour of the truth, and it will probably be not more than 18 minutes in error.

Comparing the upper and lower transit intervals of the same day with each other, we find that between the 31st May and the 13th June the lower transit intervals were the longer of the two; that after the 13th June the upper transit intervals became the larger, and remained so until the 27th June, after which the lower transit intervals again exceeded those for the upper transit.

The difference between the height of any high water and that of the low water which immediately follows it we shall hereafter call the range of the tides, also the range corresponding to the high water following the moon's upper transit we shall call the upper transit range, and that corresponding to the high water which follows the moon's lower transit, the lower transit range. Comparing the ranges of the tides for each lunar day between the 31st May and 30th June, we find that from the 31st May to 10th June, on which day the two ranges were equal, that the upper transit range was greater than the lower transit range; after the 10th June the lower transit range became the larger and so remained until the 25th June, and that after the 25th June the upper transit range again exceeded the lower transit range. Looking down the eighth column of the table, which contains the moon's declination, we observe that from the 31st May to 11th June the moon was north of the equator, which she crossed, going south between the 11th and 12th June, and remained to the south of the equator until the 25th June, when she again recrossed it into north declination.

From this we infer that when the moon is north of the equator the upper transit range will be larger than the lower transit range; but that the lower transit luni-tidal interval will be greater than the upper transit luni-tidal interval, and that when the moon is south of the equator the reverse will happen.

We will now proceed to show how the curves representing the most probable values of the luni-tidal intervals that can be derived from the observations recorded in Table I. are drawn, and how the values thus obtained are used to construct tables from which the time of high water for any given day can be calculated with sufficient accuracy for ordinary purposes.

Through the middle of an ordinary sheet of blue lined foolscap draw a straight line AB (Fig. 57) perpendicular to the blue lines, and take it to represent the mean establishment of the port, $7^h\ 35^m$; from the point A, in which AB intersects the left hand blue straight line PAQ, lay off AP and AQ, each equal to three inches; let AP represent one hour of luni-tidal interval, in the positive direction so that the point P, and all the points in a straight line drawn through it parallel to AB, will have luni-tidal intervals of $8^h\ 35^m$, and the point Q, being in the negative direction, and all the points in a straight line drawn through it parallel to AB, will have $6^h\ 35^m$ for their luni-tidal interval. Take the blue straight line next to the right of PAQ to represent that the moon's upper transit happens at 21 hours, and that her lower transit happens at 9 hours, and the blue straight line next to the right of this, or the second to the right of PAQ, to represent that the moon's upper transit happens at 22 hours, and her lower transit at 10 hours, and so on, each blue straight line in succession to the right corresponding to one hour increase in the time of the moon's transit.

Referring to Table I. we find that on the 31st May the moon's upper transit happened at $8^h\ 38^m$ A.M., or on 30th May, $20^h\ 38^m$; the high water following this transit happened at $4^h\ 25^m$ P.M. on 31st May, which gives a luni-tidal interval of $7^h\ 47^m$; subtracting from this the mean establishment $7^h\ 35^m$, we have 12 minutes for the difference. From the point A, lay off on the straight line AB in the direction of the 21 hours upper transit line, a distance equal to $\frac{19}{30}$th of that between two consecutive blue straight lines, and through the point thus determined draw a straight pencil line parallel to the blue lines. This will represent the upper transit time line $20^h\ 30^m$. From the half-inch scale, with a pair of dividers, take off 1·2, this will be equivalent to 12 minutes of luni-tidal interval; lay it off on the $20^h\ 38^m$ upper transit time line just drawn, from the point in which it cuts AB in the positive direction, and denote the point thus determined by U; consequently the point U will have for its luni-tidal interval $7^h\ 35^m + 12^m$, or $7^h\ 47^m$, and its upper transit time is $20^h\ 38^m$; an ink dot is placed at U. Proceeding in the same manner with all the upper transit high waters in the Table, we obtain the series of dots in the figure which are joined consecutively with straight pencil

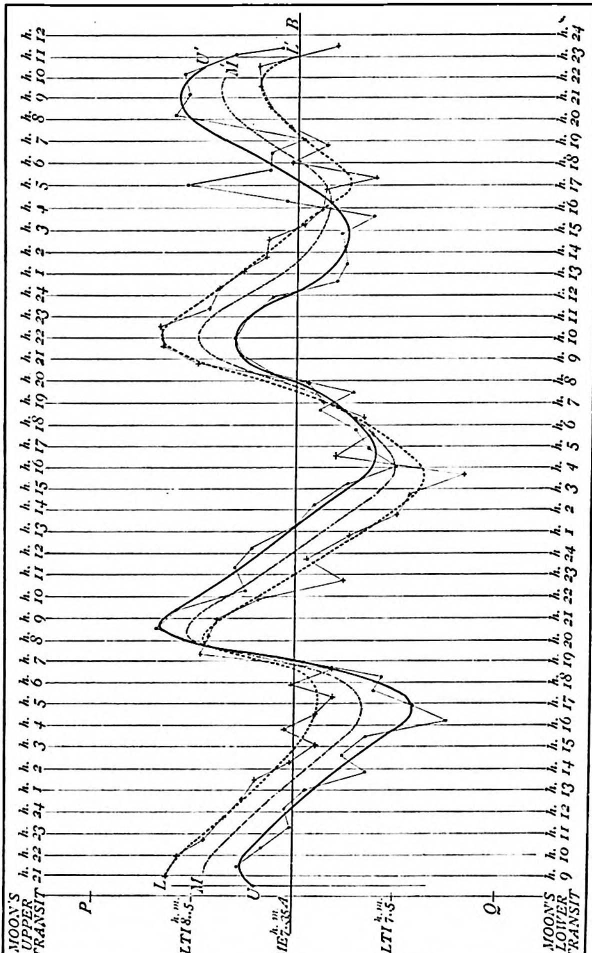

MOON'S UPPER TRANSIT
MOON'S LOWER TRANSIT
LTI 8.5
MEI 7.35A
LTI 7.5
P
Q
L
M
U
L'
M'
U'
B

FIG. 57.

lines beginning with U and the dot next to its right, and so on, and thus the multilateral figure from U to U' was drawn. Since the disturbing forces change gradually, the alterations in the luni-tidal intervals consequent on them should also be gradual; consequently the dots from U to U' inclusive ought to lie in some curve of continuous curvature. If, therefore, the observations were accurately made, and the tidal fluctuations undisturbed by extraneous forces, a curve of continuous curvature could be drawn through all the points from U to U'; the black curved line of continuous curvature is therefore drawn through, or as near as possible to, all the dots from U to U', and will give the most probable values of the upper transit intervals that can be derived from the observations.

In a similar manner, commencing with the moon's lower transit, 31st May, $9^h\ 4^m$, and luni-tidal interval $8^h\ 13^m$, we find the point L and distinguish it by a fine cross intersecting in the point, the outer ends of the cross being in ink; thus all the other crosses from L to L' representing the lower transit observations, are determined, and the dotted curve LL' of continuous curvature representing the most probable values of the lower transit luni-tidal intervals is drawn.

The remaining curve MM' was then drawn exactly half-way between the two curves UU' and LL', and represent the most probable values of the mean diurnal luni-tidal intervals that can be obtained from the observations.

The distances at which the black dots are from the UU' curve show the probable error in the observed time of high water following the moon's upper transit, which it represents; this may be due to the wind, the wash of the sea at the tide-board, from one of the small waves before mentioned coming about high water, or from the observation being carelessly taken; a similar remark will apply to the distances at which the centres of the crosses are from the dotted or LL' curve. These discrepancies serve to show the degree of dependence that we can fairly place on the tables we are about to construct, and will in a great measure account for the difference we may find between the calculated time of high water and that at which it actually occurs.

With a pair of dividers and a half-inch scale, the distances between the points in which the upper transit time lines cut respectively the MM' curve and the straight line AB are measured, and give the differences in minutes between the corresponding mean diurnal luni-tidal intervals and the mean establishment $7^h\ 35^m$. These values are inserted in Table II., the first column of which gives the time of the moon's passage across the meridian; the second column the above mentioned

differences; the third column contains the moon's horizontal parallax corresponding to her time of transit expressed in minutes; and the fourth column the moon's declination in degrees.

Table II.

GIVING THE DIFFERENCE BETWEEN THE MEAN DIURNAL LUNI-TIDAL INTERVALS AND THE MEAN ESTABLISHMENT.

Moon's Merid. Pass.	Difference in Minutes.	Moon's Horizontal Parallax.	Moon's Declination.	Moon's Merid. Pass.	Difference in Minutes.	Moon's Horizontal Parallax.	Moon's Declination.
h		′	°	h		′	°
21	+26½	58·2	11·9 N.	17	−26	59·4	0·1 N.
22	24	57·6	16·2	18	19	58·8	6·0
23	17½	57·1	18·9	19	9	58·1	11·0
24	11	56·4	20·2	20	+ 7	57·5	15·2
1	3½	55·7	20·0	21	24	56·8	18·4
2	− 4	55·2	18·2	22	30	56·2	20·0
3	11½	54·6	15·0	23	24½	55·7	20·2
4	18½	54·3	10·6	24	14½	55·1	18·8
5	19½	54·2	5·2	1	4½	54·6	15·9
6	14½	54·5	1·2 S.	2	− 2½	54·2	11·8
7	+ 6½	55·1	6·8	3	8	54·1	6·5
8	30	56·0	12·0	4	9½	54·2	0·6
9	30	57·0	16·0	5	8½	54·5	5·4 S.
10	19½	57·9	18·8	6	½	55·4	10·7
11	9	58·8	20·1	7	+10	56·3	15·1
12	0	59·4	20·0	8	19	57·4	18·3
13	− 9	59·9	18·4	9	24	58·4	19·9
14	17½	60·2	15·4	10	...	...	...
15	24½	60·1	11·2	11	...	...	...
16	29½	59·9	5·9				

Let L be the luni-tidal interval when the sun and moon are on the equator, at their mean distances from the earth, and when the high waters of the lunar and solar waves happen at the same instant; also when the moon crosses the meridian at h hours, let S_h be the difference between the time of high water of the combined wave and that of the lunar wave; let δp be the difference between the moon's horizontal parallax, and p its mean value, and let δ be the moon's declination.

The change in the luni-tidal interval of the lunar wave due to the change δp in the moon's horizontal parallax may be expressed by $A\delta p$, where A is some finite function of δp, which as far as a first approximation goes may be considered constant.

The change in the luni-tidal interval of the lunar wave resulting from the change in her declination from 0 to δ may

be expressed in a series involving the positive integral powers of cosine δ and odd positive powers of sine δ, the latter of which are eliminated from the MM' curve, leaving only the former to be accounted for. These may be represented by B versine δ, where B is some finite function of cosine δ, which we may here consider constant.

Since S_h depends on the ratio which the rate of rising or falling in the lunar wave bears to that in the solar wave, any alteration in the size of the lunar wave alters the value of S_h. Now the size of the lunar wave changes with $A\delta p$, and therefore there will be a change in S_h proportional to $A\delta p$, let this be $mAS_h\delta p$. In like manner there will be a change in S_h due to the change in the size of the lunar wave proportional to B versine δ, which will therefore be expressed by mBS_h vers δ.

Since the change in the size of the lunar wave, consequent on the change in the moon's horizontal parallax from p to $p+\delta p$, is proportional to $\frac{3\delta p}{p}$, so far as a first approximation goes, and S_h varies inversely as the size of the lunar wave, when p becomes $p+\delta p$, S_h will become $S_h\left(1-\frac{3\delta p}{p}\right)$; and if we take

$$p = 57'{\cdot}3,$$

$$S_h\left(1-\frac{3\delta p}{p}\right) = S_h(1-0{\cdot}0523\times\delta p),$$

and

$$mA = -0{\cdot}0523;$$

$$\therefore \qquad m = -\frac{0{\cdot}0523}{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(M)}$$

When therefore the moon passes the meridian at h hours, with declination δ and horizontal parallax $57'{\cdot}3+\delta p$, the mean diurnal luni-tidal interval will be expressed by

$$L_i + (A - 0{\cdot}0523\times S_h)\delta p + B \text{ versin } \delta(1+mS_h) + S_h \ldots\ldots \text{(A)}$$

if we neglect the change in the sun's declination, and its distance from the earth, during the time the observations were made, and put L_i for the mean value of L, arising from the sun being where it was, instead of on the equator and at its mean distance from the earth.

The effects of the terms involving the sine of the *sun's* declination are also eliminated from the MM' curve. We shall leave out of consideration the effects of the very small alteration in the sun's wave during the observations, and consider the fluctuations in the combined wave to be due entirely to the change in the lunar elements. We shall also consider that

the mean diurnal luni-tidal intervals corresponding to moon's meridional passages, separated by exactly twelve hours from each other, are equal, or in other words that $S_1=S_{13}$, $S_2=S_{14}$, $\ldots S_{11}=S_{23}$, and $S_{12}=S_{24}$.

To find a first approximate value for these quantities we refer to Table II., and on the fourth line from the top we find $S_{24}=11^m$, at the sixteenth line we find $S_{12}=0$. Taking the arithmetic mean of these two we have for a first approximate value of $S_0=S_{12}=S_{24}=6$. Proceeding in this manner with the other lines of the table in succession, as far as the 27th inclusive, we find the following values of S_h given in Table III., which are sufficiently accurate to be used as the multipliers of the small terms in (A).

TABLE III.

GIVING APPROXIMATE VALUES OF S_h.

Moon's Meridian Passage.	S_h in Minutes.	Moon's Meridian Passage.	S_h in Minutes.	Moon's Meridian Passage.	S_h in Minutes.
h h		h h		h h	
0 or 12	+ 6	4 or 16	− 24	8 or 20	+ 18
1 „ 13	− 3	5 „ 17	23	9 „ 21	27
2 „ 14	11	6 „ 18	17	10 „ 22	25
3 „ 15	18	7 „ 19	1	11 „ 23	17

This table can also be used to obtain a nearer approximation to the probable time of high water than that found by adding the mean establishment $7^h\ 35^m$ to the time of the moon's transit. Thus to find the time of high water when the moon passes the meridian at $2^h\ 30^m$, taking the value of $S_{2^h\,30^m}$ from Table III., we find it to be -15^m, which applied to $7^h\ 35^m$ gives $7^h\ 20^m$ for the approximate mean diurnal luni-tidal interval, which added to $2^h\ 30^m$ gives $9^h\ 50^m$ for the approximate time of high water. This will generally be sufficiently accurate for the ordinary purposes of navigation, especially when the moon is on the equator or in small declination. When her declination is large and north, eight minutes subtracted from the above will give the upper transit high water, and added will give the lower transit high water to a nearer degree of approximation, and *vice versa* when the moon's declination is large and south.

Taking the quantities from each line of Table II. in succession, commencing at the first, but omitting the last line, replacing the symbols in expression (A) by their values thus obtained, and putting $F=L_i-7^h\ 35^m$ we obtain the following thirty-six equations.

(1) $F+0{\cdot}9(A-1^{m}{\cdot}4)+0{\cdot}021\times B(1+27m)+S_{21}=26^{m}{\cdot}5$

(2) $F+0{\cdot}3(A-1{\cdot}3)+0{\cdot}040\times B(1+25m)+S_{22}=24$

(3) $F-0{\cdot}2(A-0{\cdot}9)+0{\cdot}054\times B(1+17m)+S_{23}=17{\cdot}5$

(4) $F-0{\cdot}9(A-0{\cdot}3)+0{\cdot}062\times B(1+6m)+S_{24}=11$

(5) $F-1{\cdot}6(A+0{\cdot}16)+0{\cdot}060\times B(1-3m)+S_{1}=3{\cdot}5$

(6) $F-2{\cdot}1(A+0{\cdot}57)+0{\cdot}050\times B(1-11m)+S_{2}=-4$

(7) $F-2{\cdot}7(A+0{\cdot}94)+0{\cdot}034\times B(1-18m)+S_{3}=-11{\cdot}5$

(8) $F-3{\cdot}0(A+1{\cdot}25)+0{\cdot}017\times B(1-24m)+S_{4}=-18{\cdot}5$

(9) $F-3{\cdot}1(A+1{\cdot}2)+0{\cdot}004\times B(1-23m)+S_{5}=-19{\cdot}5$

(10) $F-2{\cdot}8(A+0{\cdot}88)+S_{6}=-14{\cdot}5$

(11) $F-2{\cdot}2(A+0{\cdot}05)+0{\cdot}007\times B(1-m)+S_{7}=6{\cdot}5$

(12) $F-1{\cdot}3(A-0{\cdot}94)+0{\cdot}022\times B(1+18m)+S_{8}=30$

(13) $F-0{\cdot}3(A-1{\cdot}4)+0{\cdot}039\times B(1+27m)+S_{9}=30$

(14) $F+0{\cdot}6(A-1{\cdot}3)+0{\cdot}053\times B(1+25m)+S_{10}=19{\cdot}5$

(15) $F+1{\cdot}5(A-0{\cdot}88)+0{\cdot}061\times B(1+17m)+S_{11}=9$

(16) $F+2{\cdot}1(A-0{\cdot}31)+0{\cdot}060\times B(1+6m)+S_{12}=0$

(17) $F+2{\cdot}6(A+0{\cdot}16)+0{\cdot}051\times B(1-3m)+S_{13}=-9$

(18) $F+2{\cdot}9(A+0{\cdot}57)+0{\cdot}036\times B(1-11m)+S_{14}=-17{\cdot}5$

(19) $F+2{\cdot}8(A+0{\cdot}94)+0{\cdot}019\times B(1-18m)+S_{15}=-24{\cdot}5$

(20) $F+2{\cdot}6(A+1{\cdot}25)+0{\cdot}005\times B(1-24m)+S_{16}=-29{\cdot}5$

(21) $F+2{\cdot}1(A+1{\cdot}2)+S_{17}=-26$

(22) $F+1{\cdot}5(A+0{\cdot}88)+0{\cdot}005\times B(1-17m)+S_{18}=-19$

(23) $F+0{\cdot}8(A)+0{\cdot}018\times B(1-m)+S_{19}=-9$

(24) $F+0{\cdot}2(A-0{\cdot}9)+0{\cdot}035\times B(1+18m)+S_{20}=+7$

(25) $F-0{\cdot}5(A-1{\cdot}4)+0{\cdot}051\times B(1+27m)+S_{21}=24$

(26) $F-1{\cdot}1(A-1{\cdot}3)+0{\cdot}060\times B(1+25m)+S_{22}=30$

(27) $F-1{\cdot}6(A-0{\cdot}9)+0{\cdot}062\times B(1+17m)+S_{23}=24{\cdot}5$

(28) $F-2{\cdot}2(A-0{\cdot}31)+0{\cdot}053\times B(1+6m)+S_{24}=14{\cdot}5$

(29) $F-2{\cdot}7(A+0{\cdot}16)+0{\cdot}038\times B(1-3m)+S_{1}=4{\cdot}5$

(30) $F-3{\cdot}1(A+0{\cdot}57)+0{\cdot}021\times B(1-11m)+S_{2}=-2{\cdot}5$

(31) $F-3{\cdot}2(A+0{\cdot}94)+0{\cdot}006\times B(1-18m)+S_{3}=-8$

(32) $F-3{\cdot}1(A+1{\cdot}25)+S_{4}=-9{\cdot}5$

(33) $F-2{\cdot}8(A+1{\cdot}2)+0{\cdot}004\times B(1-23m)+S_{5}=-8{\cdot}5$

(34) $F-1{\cdot}9(A+0{\cdot}88)+0{\cdot}017\times B(1-17m)+S_{6}=-0{\cdot}5$

(35) $F-1{\cdot}0(A+\quad)+0{\cdot}035\times B(1-m)+S_{7}=10$

(36) $F-0{\cdot}1(A-0{\cdot}9)+0{\cdot}051\times B(1+18m)+S_{8}=19$

In the foregoing equations the versine moon's declination has been replaced by its numerical value to three places of decimals.

Adding equations (1) and (25),

$$2F+0{\cdot}4(A-1^{\mathrm{m}}{\cdot}4)+0{\cdot}072\times B(1+27m)+2S_{21}=50^{\mathrm{m}}{\cdot}5.$$

Multiplying (13) by 2,

$$2F-0{\cdot}6(A-1^{\mathrm{m}}{\cdot}4)+0{\cdot}078\times B(1+27m)+2S_{9}\;=60.$$

Subtracting the latter of these equations from the former and putting $S_9=S_{21}$, we have,

(I) $$A-0{\cdot}006\times B(1+27m)=-\;\;8^{\mathrm{m}}{\cdot}1$$

In like manner (2)+(26)−2(14) give

(II) $$2A+0{\cdot}006\times B(1+25m)=-12\;{\cdot}4$$

" (3)+(27)−2(15) give

(III) $$4{\cdot}8\times A+0{\cdot}006\times B(1+17m)=-19\;{\cdot}8$$

" (4)+(28)−2(16) give

(IV) $$7{\cdot}3\times A+0{\cdot}005\times B(1+\;\;6m)=-23\;{\cdot}2$$

" (5)+(29)−2(17) give

(V) $$9{\cdot}5\times A+0{\cdot}004\times B(1-\;\;3m)=-27\;{\cdot}5$$

" (6)+(30)−2(18) give

(VI) $$11A+0{\cdot}001\times B(1-11m)=-34\;{\cdot}8$$

" (7)+(31)−2(19) give

(VII) $$11{\cdot}5\times A-0{\cdot}002\times B(1-18m)=-40\;{\cdot}3$$

" (8)+(32)−2(20) give

(VIII) $$11{\cdot}3\times A-0{\cdot}007\times B(1-24m)=-45\;{\cdot}1$$

" (9)+(33)−2(21) give

(IX) $$10{\cdot}1\times A-0{\cdot}008+B(1-23m)=-36\;{\cdot}1$$

" (10)+(34)−2(22) give

(X) $$7{\cdot}7\times A-0{\cdot}007\times B(1-17m)=-29\;{\cdot}8$$

" (11)+(35)−2(23) give

(XI) $$4{\cdot}9\times A-0{\cdot}006\times B(1-\quad m)=-39\;{\cdot}4$$

" (12)+(36)−2(24) give

(XII) $$1{\cdot}6\times A-0{\cdot}003\times B(1+18m)=-33\;{\cdot}7.$$

Omitting equations (I) and (XII) in which the co-efficients of A are too small to make them important, besides the disadvantage of their having the same sign to the co-efficients of B; adding the others, two and two, so as to make the terms involving B so small as to be practically of no importance for the determination of A; and omitting them, we find that

(II)+(XI) give	$6{\cdot}9 \times A = -41^{m}{\cdot}8$	(XIII)
(III)+(X)	$12{\cdot}5 \times A = -49\ {\cdot}6$	(XIV)
(IV)+(IX)	$17{\cdot}4 \times A = -59\ {\cdot}3$	(XV)
(V)+(VIII)	$20{\cdot}8 \times A = -72\ {\cdot}6$	(XVI)
(VI)+(VII)	$22{\cdot}5 \times A = -75\ {\cdot}1$	(XVII)

Combining these as nearly in proportion to their respective coefficients of A, as small whole numbers can be found, we have

(XIII) multiplied by	3 gives	$20{\cdot}7 \times A = -\ 125^{m}{\cdot}4,$
(XIV)	6	$75{\cdot}0 \times A = -\ 297\ {\cdot}6,$
(XV)	9	$156{\cdot}6 \times A = -\ 533\ {\cdot}7,$
(XVI)	10	$208{\cdot}0 \times A = -\ 726\ {\cdot}0,$
(XVII)	11	$247{\cdot}5 \times A = -\ 826\ {\cdot}1.$
Adding these,		$707{\cdot}8 \times A = -2508\ {\cdot}8.$

$$A = -\frac{2508^{m}{\cdot}8}{707{\cdot}8} = -3^{m}{\cdot}684 \quad \ldots\ldots\ldots\ldots (a)$$

Referring to equation (M), we find

$$m = -\frac{0{\cdot}0523}{A} = \frac{0{\cdot}0523}{3{\cdot}684}$$
$$= 0{\cdot}014 \quad \ldots\ldots\ldots\ldots (\mu)$$

We will now proceed to determine the best value of B that can be derived from the foregoing equations. The coefficients of B being all positive, and the difference in the moon's declination for the same hours of the moon's meridian passage not being generally of sufficient size, we cannot obtain so satisfactory a value of B as if the observations had extended over a longer period, but still they will give a fair value of B and suffice to show the mode of procedure.

Subtracting equation (1) from (25), we have

$$-1{\cdot}4 \times (A - 1^{m}{\cdot}4) + 0{\cdot}030 \times B(1 + 27m) = -2^{m}{\cdot}5$$

Introducing the values of A and m, as above determined,

$$1{\cdot}4 \times 5^{m}{\cdot}08 + 0{\cdot}030 \times B \times 1{\cdot}38 = 2^{m}{\cdot}5$$
$$\therefore\ 0{\cdot}041 \times B = -9^{m}{\cdot}6 \quad \ldots\ldots\ldots\ldots (\text{XIX})$$

Equations (2) and (26) in the same way give

$$0{\cdot}027 \times B = -1 \quad \ldots\ldots\ldots\ldots (\text{XX})$$

Equations (3) and (27) give

$$0{\cdot}010 \times B = \quad 0{\cdot}6 \quad \ldots\ldots\ldots\ldots (\text{XXI})$$

Equations (29) and (5) give	$0{\cdot}021 \times B = 2{\cdot}5$	....	(XXII)
(30) (6)	$0{\cdot}016 \times B = 1{\cdot}6$	....	(XXIII)
(31) (7)	$0{\cdot}021 \times B = -2{\cdot}1$	....	(XXIV)
(32) (8)	$0{\cdot}011 \times B = -8{\cdot}7$	....	(XXV)
(10) (34)	$0{\cdot}013 \times B = 16{\cdot}6$	....	(XXVI)
(11) (35)	$0{\cdot}028 \times B = 7{\cdot}9$	....	(XXVII)
(12) (36)	$0{\cdot}036 \times B = -5{\cdot}5$	....	(XXVIII)
(28) (4)	$0{\cdot}010 \times B = 1{\cdot}7$	....	(XXIX)

Giving these their probable values,

(XIX) multiplied by	4 gives	$0{\cdot}164 \times B = -38^{m}{\cdot}4$
(XX)	3	$0{\cdot}081 \times B = -\ 3\ {\cdot}0$
(XXI)	1	$0{\cdot}010 \times B = +\ 0\ {\cdot}6$
(XXII)	2	$0{\cdot}042 \times B = +\ 5\ {\cdot}0$
(XXIII)	2	$0{\cdot}032 \times B = +\ 3\ {\cdot}2$
(XXIV)	2	$0{\cdot}042 \times B = -\ 4\ {\cdot}2$
(XXV)	1	$0{\cdot}011 \times B = -\ 8\ {\cdot}7$
(XXVI)	1	$0{\cdot}013 \times B = +16\ {\cdot}6$
(XXVII)	3	$0{\cdot}084 \times B = +23\ {\cdot}7$
(XXVIII)	4	$0{\cdot}144 \times B = -22\ {\cdot}0$
(XXIX)	1	$0{\cdot}010 \times B = \ 1\ {\cdot}7$

Adding these, $0{\cdot}633 \times B = -27^{m}{\cdot}2$

$$\therefore B = -\frac{27200^{m}}{633} = -43^{m} \text{...........} (\beta)$$

Substituting A, B, and m in the 36 foregoing equations (1), (2)...(35) and (36) by their values as given in equations (α), (β), and (μ), and carrying the results to the right hand side of each equation respectively, from equation

(1) we have	$F+S_{21} =$	$26^{m}{\cdot}5 + 4^{m}{\cdot}6 + 1^{m}{\cdot}3 =$	$32^{m}{\cdot}4$
(2)	$F+S_{22} =$	$24 \quad +1\ {\cdot}5 + 2\ {\cdot}1 =$	$27\ {\cdot}6$
(3)	$F+S_{23} =$	$17\ {\cdot}5 - 1\ {\cdot}4 + 2\ {\cdot}9 =$	19
(4)	$F+S_{24} =$	$11 \quad -3\ {\cdot}6 + 2\ {\cdot}9 =$	$10\ {\cdot}3$
(5)	$F+S_{1} =$	$3\ {\cdot}5 - 5\ {\cdot}6 + 2\ {\cdot}6 =$	$0\ {\cdot}5$
(6)	$F+S_{2} =$	$-\ 4 \quad -6\ {\cdot}5 + 1\ {\cdot}8 =$	$-\ 8\ {\cdot}7$
(7)	$F+S_{3} =$	$-11\ {\cdot}5 - 7\ {\cdot}3 + 1\ {\cdot}1 =$	$-17\ {\cdot}5$
(8)	$F+S_{4} =$	$-18\ {\cdot}5 - 7\ {\cdot}2 + 0\ {\cdot}5 =$	$-25\ {\cdot}2$
(9)	$F+S_{5} =$	$-19\ {\cdot}5 - 7\ {\cdot}7 + 0\ {\cdot}1 =$	$-27\ {\cdot}1$
(10)	$F+S_{6} =$	$-\ 14\ {\cdot}5 - 7\ {\cdot}8$	$= -22\ {\cdot}3$
(11)	$F+S_{7} =$	$6\ {\cdot}5 - 7\ {\cdot}9 + 0\ {\cdot}3 =$	$-\ 1\ {\cdot}1$

(12)	$F+S_8 = 30{\cdot}0-6{\cdot}0+1{\cdot}2=+25{\cdot}2$
(13)	$F+S_9 = 30{\cdot}0-1{\cdot}5+2{\cdot}3=+30{\cdot}8$
(14)	$F+S_{10}= 19{\cdot}5+3{\cdot}0+3{\cdot}1=+25{\cdot}6$
(15)	$F+S_{11}= 9{\cdot}0+6{\cdot}7+3{\cdot}2=+18{\cdot}9$
(16)	$F+S_{12}= 0+8{\cdot}4+2{\cdot}8=+11{\cdot}2$
(17)	$F+S_{13}=-9{\cdot}0+9{\cdot}1+2{\cdot}2=+2{\cdot}3$
(18)	$F+S_{14}=-17{\cdot}5+9{\cdot}0+1{\cdot}3=-7{\cdot}2$
(19)	$F+S_{15}=-24{\cdot}5+7{\cdot}6+0{\cdot}6=-16{\cdot}3$
(20)	$F+S_{16}=-29{\cdot}0+6{\cdot}2+0{\cdot}1=-23{\cdot}2$
(21)	$F+S_{17}=-26{\cdot}0+5{\cdot}2=-20{\cdot}8$
(22)	$F+S_{18}=-19{\cdot}0+4{\cdot}2+0{\cdot}1=-14{\cdot}7$
(23)	$F+S_{19}=-9{\cdot}0+3{\cdot}0+0{\cdot}8=-5{\cdot}2$
(24)	$F+S_{20}=+7{\cdot}0+0{\cdot}9+1{\cdot}8=+9{\cdot}7$
(25)	$F+S_{21}=+24{\cdot}0-2{\cdot}5+3{\cdot}0=+24{\cdot}5$
(26)	$F+S_{22}=+30{\cdot}5-5{\cdot}5+3{\cdot}0=+27{\cdot}5$
(27)	$F+S_{23}=+24{\cdot}5-7{\cdot}4+3{\cdot}0=+20{\cdot}1$
(28)	$F+S_{24}=+30{\cdot}5-8{\cdot}8+2{\cdot}5=+8{\cdot}2$
(29)	$F+S_1=+4{\cdot}5-9{\cdot}4+1{\cdot}5=-3{\cdot}4$
(30)	$F+S_2=-2{\cdot}5-9{\cdot}6+0{\cdot}8=-11{\cdot}3$
(31)	$F+S_3=-8{\cdot}0-8{\cdot}6+0{\cdot}2=-16{\cdot}4$
(32)	$F+S_4=-9{\cdot}5-7{\cdot}4=-16{\cdot}9$
(33)	$F+S_5=-8{\cdot}5-7{\cdot}0+0{\cdot}1=-15{\cdot}4$
(34)	$F+S_6=-0{\cdot}5-5{\cdot}3+0{\cdot}5=-5{\cdot}3$
(35)	$F+S_7=+10{\cdot}0-3{\cdot}7+1{\cdot}5=+7{\cdot}8$
(36)	$F+S_8=+19{\cdot}0-0{\cdot}3+2{\cdot}7=+21{\cdot}4$

Taking the arithmetic mean of (1), (13), and (25), regarding S_9, as equal to S_{21}, we have $F+S_9 = 29^m{\cdot}2$...(37)

Similarly, (2), (14), and (26) give $F+S_{10}= 26{\cdot}9$...(38)

(3),	(15),	(27)	$F+S_{11}= 19{\cdot}3$	...(39)
(4),	(16),	(28)	$F+S_{12}= 9{\cdot}9$	...(40)
(5),	(17),	(29)	$F+S_1=-0{\cdot}2$	...(41)
(6),	(18),	(30)	$F+S_2=-9{\cdot}1$	...(42)
(7),	(19),	(31)	$F+S_3=-16{\cdot}8$	...(43)
(8),	(20),	(32)	$F+S_4=-21{\cdot}8$	...(44)
(9),	(21),	(33)	$F+S_5=-21{\cdot}1$	...(45)
(10),	(22),	(34)	$F+S_6=-14{\cdot}1$	...(46)
(11),	(23),	(35)	$F+S_7=+0{\cdot}5$	...(47)
(12),	(24),	(36)	$F+S_8=+18{\cdot}9$	...(48)

Adding these and putting $S_9+S_{10}+\ldots+S_7+S_8=0$,

we have $$12F=21^{m}\cdot6$$

$$\therefore F=\ \ 1\ \cdot8 \ldots\ldots\ldots\ldots(\phi)$$

and $$L_{\iota}=7^{h}\ 35^{m}+F=7^{h}\ 36^{m}\cdot8 \ldots\ldots\ldots\ldots(\pi)$$

and equations (41), (42), ... (47), (48), (37), ... (40) give the values of S_1 and S_{12} inserted in the following table.

Table IV.

GIVING THE VALUES OF S_h WHEN THE MOON'S HORIZONTAL PARALLAX IS 57′·3.

Moon's Meridian Passage.	S_h in Minutes.	Moon's Meridian Passage.	S_h in Minutes.	Moon's Meridian Passage.	S_h in Minutes.	Moon's Meridian Passage.	S_h in Minutes.
h		h		h		h	
1	− 2·0	4	−23·6	7	− 1·3	10	+25·1
2	−10·9	5	−22·9	8	+17·1	11	+17·5
3	−18·6	6	−15·9	9	+27·4	12	+ 8·1

The correction to the moon's luni-tidal interval for each hour of her meridian passage due to an increase of one minute of angle in her horizontal parallax can now be calculated; they are inserted in Table V., as well as the corresponding values of $1+mS_h$.

Table V.

THE CORRECTION TO THE LUNI-TIDAL INTERVAL DUE TO ONE MINUTE (1′) INCREASE IN THE MOON'S HORIZONTAL PARALLAX, AS WELL AS THE VALUES OF $1+mS_h$.

Moon's Meridian Passage.	Correction.	Values of $1+mS_h$.	Moon's Meridian Passage.	Correction.	Values of $1+mS_h$.
1^h A.M. or P.M.	−3 ·57	0·97	7^h A.M. or P.M.	$-3^m\cdot60$	0·98
2 " "	3 ·13	0·85	8 " "	4 ·57	1·24
3 " "	2 ·72	0·74	9 " "	5 ·07	1·38
4 " "	2 ·47	0·67	10 " "	4 ·96	1·35
5 " "	2 ·50	0·68	11 " "	4 ·57	1·24
6 " "	2 ·87	0·78	12 " "	4 ·09	1·11

Upon examining the observations recorded in Table I., it will be seen, as previously mentioned, that, when the moon's declination is north, the superior transit wave is larger than the inferior transit wave, but the luni-tidal interval of the former is

smaller than that of the latter. We assume, therefore, that when $S_h = 0$, or when the lunar and solar high waters happen at the same time, that the difference between the two intervals may be expressed by $2\left(\frac{p}{57 \cdot 3}\right)^3 M \sin \delta$, where M is some function of the even powers of $\sin d$, which may be regarded as constant. We have also found that when $S_h = 0$, an alteration in the size of the lunar wave, which causes an alteration A in the luni-tidal interval, causes an alteration $A(1 + mS_h)$ in the interval when S_h has any other value, and $m = 0{\cdot}014$. We therefore assume that the difference between the superior transit and inferior transit luni-tidal intervals, when the moon's horizontal parallax is p, and her declination δ, will be all represented by

$$2\left(\frac{p}{57 \cdot 3}\right)^3 M \sin \delta (1 + 0{\cdot}014 \times S_h) \ldots\ldots\ldots\ldots\ldots\ldots \text{(D)}$$

During the observations we are discussing, the sun's declination averaged about 23° N., and the diurnal inequality of the solar waves was consequently large, and produced a sensible effect on the diurnal inequality of the luni-tidal intervals, of which it is necessary to take account. Again referring to Table I., we at once perceive that the higher of the two high waters which happen on the same day is followed by the lower of the two low waters, and that the distance which each tide has to rise, from low water to the high water which immediately *follows*, is very nearly the same for both the waves; whilst the fall from the higher high water to the lower low water is much larger than the fall from the lower high water to the higher low water which follows it. Assuming that the character of the solar wave is similar to that of the combined wave, the velocity of rising in the two solar tides of the same day will be so nearly equal as to produce effects in the luni-tidal intervals, which differ so slightly from each other that they will not sensibly alter their difference; but the velocity of falling in the two waves being unequal, will unequally affect the two luni-tidal intervals and show itself in their difference, which may therefore be represented by $2sS_h$, where s depends on the sine of the sun's declination. Table IV. shows that S_h is very small when the moon passes the meridian at 1^h and 7^h; from which we infer that the solar tidal interval is about one hour larger than the luni-tidal intervals, and that the sun's tide will be falling at the time of high water when the moon passes the meridian at any time from 2^h to 6^h inclusive, either A.M. or P.M.; and consequently when the moon passes the meridian at these times, the term $2sS_h$ must be introduced, Table IV. furnishing the values of S_h for each hour of the moon's meridian passage.

Table V.

GIVING THE EXPRESSIONS FOR THE DIURNAL INEQUALITIES OF THE LUNI-TIDAL INTERVALS.

Moon's Meridian Passage.	Expression for moon's declination N., and her upper transit.
1^h P.M., - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 M \sin\delta.$
2 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 0\cdot85\times M\sin\delta - 22s$
3 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 0\cdot74\times M\sin\delta - 37s$
4 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 0\cdot67\times M\sin\delta - 47s$
5 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 0\cdot68\times M\sin\delta - 46s$
6 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 0\cdot78\times M\sin\delta - 32s$
7 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 M\sin\delta$
8 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 1\cdot24\times M\sin\delta$
9 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 1\cdot38\times M\sin\delta$
10 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 1\cdot35\times M\sin\delta$
11 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 1\cdot24\times M\sin\delta$
12 „ - - -	$-2\left(\frac{p}{57\cdot3}\right)^3 1\cdot11\times M\sin\delta$

When the moon's upper transit is A.M. the signs of the terms involving s in the above table must be changed. When the moon's declination is south, the signs of the terms involving M must be changed.

The diurnal inequality in the luni-tidal interval for each hour of the moon's meridian passage during the period of observation are found from Fig. 52, by measuring the length of the part of each time line intercepted between the points in which they are intersected respectively by the black and dotted curves; these lengths in half inches give the minutes of the diurnal inequalities. In this way and replacing p and δ by their

values in Table II. we have the following equations after dividing by 2.

For moon's meridian passage—

$$21^h \quad \left(\frac{58\cdot2}{57\cdot3}\right)^3 M \sin(11^\circ\cdot9) \times 1\cdot38 = 11^m\cdot5 \quad \ldots (1)$$

$$22 \quad \left(\frac{57\cdot6}{57\cdot3}\right)^3 1\cdot35 \times M \sin(16^\circ\cdot2) = 10\cdot0 \quad \ldots (2)$$

$$23 \quad \left(\frac{57\cdot1}{57\cdot3}\right)^3 \times 1\cdot24 \times M \sin(18^\circ\cdot9) = 9\cdot25 \ldots (3)$$

$$24 \quad \left(\frac{56\cdot4}{57\cdot3}\right)^3 1\cdot11 \times M \sin(20^\circ\cdot2) = 8\cdot5 \quad \ldots (4)$$

$$1 \quad \left(\frac{55\cdot7}{57\cdot3}\right)^3 M \sin(20^\circ) = 9 \quad \ldots (5)$$

$$2 \quad \left(\frac{55\cdot2}{57\cdot3}\right)^3 0\cdot85 \times M \sin(18^\circ) + 11s = 9\cdot75 \ldots (6)$$

$$3 \quad \left(\frac{54\cdot6}{57\cdot3}\right)^3 0\cdot74 \times M \sin(15^\circ) + 18\cdot5 \times s = 11\cdot0 \quad \ldots (7)$$

$$4 \quad \left(\frac{54\cdot3}{57\cdot3}\right)^3 0\cdot67 \times M \sin(10^\circ\cdot6) + 23\cdot5 \times s = 13\cdot75 \ldots (8)$$

$$5 \quad \left(\frac{54\cdot2}{57\cdot3}\right)^3 0\cdot68 \times M \sin(5^\circ\cdot2) + 23s = 13\cdot75 \ldots (9)$$

$$6 \quad \left(\frac{54\cdot5}{57\cdot3}\right)^3 0\cdot78 \times M \sin(1^\circ\cdot2) - 16s = -9\cdot0 \quad \ldots (10)$$

$$7 \quad \left(\frac{55\cdot1}{57\cdot3}\right)^3 M \sin(6^\circ\cdot8) = -4\cdot5 \quad \ldots (11)$$

$$8 \quad \left(\frac{56\cdot0}{57\cdot3}\right)^3 1\cdot24 \times M \sin(12^\circ) = 3\cdot25 \ldots (12)$$

$$9 \quad \left(\frac{57\cdot0}{57\cdot3}\right)^3 1\cdot38 \times M \sin(16^\circ) = 8\cdot25 \ldots (13)$$

$$10 \quad \left(\frac{57\cdot9}{57\cdot3}\right)^3 1\cdot35 \times M \sin(18^\circ\cdot8) = 9\cdot0 \quad \ldots (14)$$

$$11 \quad \left(\frac{58\cdot8}{57\cdot3}\right)^3 1\cdot24 \times M \sin(20^\circ\cdot1) = 9\cdot25 \ldots (15)$$

$$12 \quad \left(\frac{59\cdot4}{57\cdot3}\right)^3 1\cdot11 \times M \sin(20^\circ) = 9\cdot75 \ldots (16)$$

$$13 \quad \left(\frac{59\cdot9}{57\cdot3}\right)^3 M \sin(18^\circ\cdot4) = 11 \quad \ldots (17)$$

$$14 \quad \left(\frac{60\cdot2}{57\cdot3}\right)^3 \times 0\cdot85 \times M \sin(15\cdot4) + 11s = 11\cdot5 \quad \ldots (18)$$

15^h	$\left(\frac{60\cdot1}{57\cdot3}\right)^3 \times 0\cdot74 \times M \sin(11^\circ\cdot2) + 18\cdot5 \times s$	$= 10 \cdot5$	...(19)
16	$\left(\frac{59\cdot9}{57\cdot3}\right)^3 \times 0\cdot67 \times M \sin(5^\circ\cdot9) + 23\cdot5 \times s$	$= 7$	...(20)
17	$\left(\frac{59\cdot4}{57\cdot3}\right)^3 \times 0\cdot68 \times M \sin(0^\circ\cdot1) - 23s$	$= 3 \cdot25$	...(21)
18	$\left(\frac{58\cdot8}{57\cdot3}\right)^3 \times 0\cdot78 \times M \sin(6^\circ) - 16s$	$= -1$	...(22)
19	$\left(\frac{58\cdot1}{57\cdot3}\right)^3 M \sin(11^\circ)$	$= 1$	...(23)
20	$\left(\frac{57\cdot5}{57\cdot3}\right)^3 1\cdot24 \times M \sin(15^\circ\cdot2)$	$= 4$	...(24)
21	$\left(\frac{56\cdot8}{57\cdot3}\right)^3 1\cdot38 \times M \sin(18^\circ\cdot4)$	$= 8$	...(25)
22	$\left(\frac{56\cdot2}{57\cdot3}\right)^3 1\cdot35 \times M \sin(20^\circ\cdot0)$	$= 11$	...(26)
23	$\left(\frac{55\cdot7}{57\cdot3}\right)^3 1\cdot24 \times M \sin(20^\circ\cdot2)$	$= 10$	...(27)
24	$\left(\frac{55\cdot1}{57\cdot3}\right)^3 1\cdot11 \times M \sin(18^\circ\cdot8)$	$= 11 \cdot5$	...(28)
1	$\left(\frac{54\cdot6}{57\cdot3}\right)^3 M \sin(15^\circ\cdot9)$	$= 12$	...(29)
2	$\left(\frac{54\cdot2}{57\cdot3}\right)^3 \times 0\cdot85 \times M \sin(11^\circ\cdot8) + 11s$	$= 11$	...(30)
3	$\left(\frac{54\cdot1}{57\cdot3}\right)^3 \times 0\cdot74 \times M \sin(6^\circ\cdot5) + 18\cdot5 \times s$	$= 7 \cdot5$	...(31)
4	$\left(\frac{54\cdot3}{57\cdot3}\right)^3 \times 0\cdot67 \times M \sin(0^\circ\cdot6) + 23\cdot5 \times s$	$= 2$	...(32)
5	$\left(\frac{54\cdot5}{57\cdot3}\right)^3 \times 0\cdot68 \times M \sin(5^\circ\cdot4) - 23s$	$= 6 \cdot5$	...(33)
6	$\left(\frac{55\cdot4}{57\cdot3}\right)^3 \times 0\cdot78 \times M \sin(10^\circ\cdot7) - 16s$	$= 9$	...(34)
7	$\left(\frac{56\cdot3}{57\cdot3}\right)^3 M \sin(15^\circ\cdot1)$	$= 12 \cdot25$	...(35)
8	$\left(\frac{57\cdot4}{57\cdot3}\right)^3 \times 24 \times M \sin(18^\circ\cdot3)$	$= 13 \cdot25$	...(36)
9	$\left(\frac{58\cdot4}{57\cdot3}\right)^3 \times 1\cdot38 \times M \sin(19^\circ\cdot9)$	$= 12 \cdot5$	...(37)

To calculate the coefficients of M three places of logarithms will be sufficient.

For equation (1)	log 58·2, - - -	1·765
	log 57·3, - - -	1·758
		0·007
		3
		0·021
	log 1·38, - - -	0·140
	log sin(11°·9), - -	9·314
	log 0·30, - - -	$\bar{1}$·475

Proceeding in this manner with the other equations we find as follows.

Equation (1) becomes	$0{\cdot}30 \times M$	$= 11^{m}{\cdot}5$
(2)	$0{\cdot}38 \times M$	$= 10$
(3)	$0{\cdot}40 \times M$	$= 9{\cdot}25$
(4)	$0{\cdot}37 \times M$	$= 8{\cdot}5$
(5)	$0{\cdot}31 \times M$	$= 9$
(6)	$0{\cdot}24 \times M + 11s$	$= 9{\cdot}75$
(7)	$0{\cdot}17 \times M + 18{\cdot}5 \times s$	$= 11$
(8)	$0{\cdot}11 \times M + 23{\cdot}5 \times s$	$= 13{\cdot}75$
(9)	$0{\cdot}06 \times M + 23s$	$= 13{\cdot}75$
(10)	$0{\cdot}02 \times M - 16s$	$= -9$
(11)	$0{\cdot}11 \times M$	$= -4{\cdot}5$
(12)	$0{\cdot}24 \times M$	$= 3{\cdot}75$
(13)	$0{\cdot}38 \times M$	$= 8{\cdot}25$
(14)	$0{\cdot}45 \times M$	$= 9$
(15)	$0{\cdot}46 \times M$	$= 9{\cdot}25$
(16)	$0{\cdot}42 \times M$	$= 9{\cdot}75$
(17)	$0{\cdot}36 \times M$	$= 11$
(18)	$0{\cdot}26 \times M + 11s$	$= 11{\cdot}5$
(19)	$0{\cdot}15 \times M + 18{\cdot}5 \times s$	$= 10{\cdot}5$
(20)	$0{\cdot}08 \times M + 23{\cdot}5 \times s$	$= 7$
(21)	$23s$	$= 3{\cdot}75$
(22)	$0{\cdot}09 \times M - 16s$	$= -1$
(23)	$0{\cdot}20 \times M$	$= 1$
(24)	$0{\cdot}33 \times M$	$= 4$
(25)	$0{\cdot}42 \times M$	$= 8$
(26)	$0{\cdot}43 \times M$	$= 11$
(27)	$0{\cdot}39 \times M$	$= 10$
(28)	$0{\cdot}32 \times M$	$= 11{\cdot}5$
(29)	$0{\cdot}24 \times M$	$= 12$
(30)	$0{\cdot}15 \times M + 11s$	$= 11$

Equation (31) becomes	$0{\cdot}07 \times M + 18{\cdot}5 \times s$	$= 7{\cdot}5$
(32)	$+ 23{\cdot}5 \times s$	$= 2$
(33)	$0{\cdot}06 \times M - 23s$	$= 6{\cdot}5$
(34)	$0{\cdot}13 \times M - 16s$	$= 9$
(35)	$0{\cdot}25 \times M$	$= 12{\cdot}25$
(36)	$0{\cdot}39 \times M$	$= 13{\cdot}25$
(37)	$0{\cdot}50 \times M$	$= 12{\cdot}5$

From these 37 equations to obtain probable values of M and s which on the average will best satisfy them, we proceed as follows.

Multiply equation (1) by	3	$0{\cdot}90 \times M$	$= 34^{m}{\cdot}5$
(2)	4	$1{\cdot}52 \times M$	$= 40$
(3)	4	$1{\cdot}60 \times M$	$= 37$
(4)	4	$1{\cdot}48 \times M$	$= 34$
(5)	3	$0{\cdot}93 \times M$	$= 27$
(6)	2	$0{\cdot}48 \times M + 22s$	$= 19{\cdot}5$
(7)	2	$0{\cdot}34 \times M + 37s$	$= 22$
(8)	1	$0{\cdot}11 \times M + 23{\cdot}5 \times s$	$= 13{\cdot}75$
(9)	1	$0{\cdot}06 \times M + 23s$	$= 13{\cdot}75$
(11)	1	$0{\cdot}11 \times M$	$= -\ 4{\cdot}5$
(12)	2	$0{\cdot}48 \times M$	$= 7{\cdot}5$
(13)	4	$1{\cdot}52 \times M$	$= 33$
(14)	4	$1{\cdot}80 \times M$	$= 36$
(15)	5	$2{\cdot}30 \times M$	$= 46{\cdot}25$
(16)	4	$1{\cdot}68 \times M$	$= 39$
(17)	4	$1{\cdot}44 \times M$	$= 44$
(18)	3	$0{\cdot}78 \times M + 33s$	$= 34{\cdot}5$
(19)	1	$0{\cdot}15 \times M + 18{\cdot}5 \times s$	$= 10{\cdot}5$
(20)	1	$0{\cdot}08 \times M + 23{\cdot}5 \times s$	$= 7$
(22)	1	$0{\cdot}09 \times M - 16s$	$= -\ 1$
(23)	2	$0{\cdot}40 \times M$	$= 2$
(24)	3	$0{\cdot}99 \times M$	$= 12$
(25)	4	$1{\cdot}68 \times M$	$= 32$
(26)	4	$1{\cdot}72 \times M$	$= 44$
(27)	4	$1{\cdot}56 \times M$	$= 40$
(28)	3	$0{\cdot}96 \times M$	$= 34{\cdot}5$
(29)	2	$0{\cdot}48 \times M$	$= 24$
(30)	1	$0{\cdot}15 \times M + 11s$	$= 11$
(31)	1	$0{\cdot}07 \times M + 18{\cdot}5 \times s$	$= 7{\cdot}5$
(33)	1	$0{\cdot}06 \times M - 23s$	$= 6{\cdot}5$
(34)	1	$0{\cdot}13 \times M - 16s$	$= 9$
(35)	2	$0{\cdot}50 \times M$	$= 24{\cdot}5$
(36)	4	$1{\cdot}56 \times M$	$= 53$
(37)	5	$2{\cdot}50 \times M$	$= 62{\cdot}5$

Adding these,

$$30{\cdot}61 \times M + 155 \times s = 856{\cdot}75 \ldots\ldots\ldots\ldots\ldots\ldots (\mu)$$

which furnishes the equation giving a good average of M. To obtain a good equation in s we proceed as follows.

$$\begin{array}{lrl} \text{Equation (6) gives} & 11s + 0{\cdot}24 \times M = & 9^{m}{\cdot}75 \\ \quad (18) & 11s + 0{\cdot}26 \times M = & 11{\cdot}5 \\ \quad (30) & 11s + 0{\cdot}15 \times M = & 11 \\ \text{adding,} & 33s + 0{\cdot}65 \times M = & 32{\cdot}25 \ldots\ldots (\sigma_1) \end{array}$$

$$\begin{array}{lrl} \text{Equation (7) gives} & 18{\cdot}5 \times s + 0{\cdot}17 \times M = & 11 \\ \quad (19) & 18{\cdot}5 \times s + 0{\cdot}15 \times M = & 10{\cdot}15 \\ \quad (31) & 18{\cdot}5 \times s + 0{\cdot}07 \times M = & 7{\cdot}5 \\ \text{adding,} & 55{\cdot}5 \times s + 0{\cdot}39 \times M = & 29 \ldots\ldots (\sigma_2) \end{array}$$

$$\begin{array}{lrl} \text{Equation (8) gives} & 23{\cdot}5 \times s + 0{\cdot}11 \times M = & 13{\cdot}75 \\ \quad (20) & 23{\cdot}5 \times s + 0{\cdot}08 \times M = & 7 \\ \quad (32) & 23{\cdot}5 \times s \quad = & 2 \\ \text{adding,} & 70{\cdot}5 \times s + 0{\cdot}19 \times M = & 22{\cdot}75 \ldots\ldots (\sigma_3) \end{array}$$

$$\begin{array}{lrl} \text{Equation (9) gives} & 23s + 0{\cdot}06 \times M = & 13{\cdot}75 \\ \quad (21) & 23s \quad = & 3{\cdot}75 \\ \quad (33) & 23s - 0{\cdot}06 \times M = & -\ 6{\cdot}5 \\ \text{adding,} & 69s \quad = & 11 \ldots\ldots (\sigma_4) \end{array}$$

$$\begin{array}{lrl} \text{Equation (10) gives} & 16s - 0{\cdot}02 \times M = & 9 \\ \quad (22) & 16s - 0{\cdot}09 \times M = & 1 \\ \quad (34) & 16s - 0{\cdot}13 \times M = & -\ 9 \\ \text{adding,} & 48s - 0{\cdot}24 \times M = & 1 \ldots\ldots (\sigma_5) \end{array}$$

$$\begin{array}{lrl} \sigma_1 \times \frac{4}{3} \text{ gives} & 44s + 0{\cdot}87 \times M = & 43 \\ \sigma_2 \times \frac{7}{3} & 129{\cdot}5 \times s + 0{\cdot}91 \times M = & 67{\cdot}7 \\ \sigma_5 \times 2 & 96 \times s - 0{\cdot}48 \times M = & 2 \\ \text{adding,} & 269{\cdot}5 \times s + 1{\cdot}30 \times M = & 112{\cdot}7 \\ \text{dividing by 6,} & 44{\cdot}9 \times s + 0{\cdot}217 \times M = & 18{\cdot}78 \ldots\ldots (\sigma_6) \end{array}$$

equivalent in value to nine equations, with coefficient of $s = 45$;

Adding $\sigma_3 \times \sigma_4$,

$$139{\cdot}5 \times \sigma + 0{\cdot}19 \times M = 33{\cdot}75;$$

dividing by 2, $\quad 69{\cdot}75 \times \sigma + 0{\cdot}095 \times M = 16{\cdot}875 \ldots\ldots\ldots\ldots\ldots (\sigma_7)$

equivalent to six equations, with coefficient s of 70; hence (σ_6) and (σ_7) are so nearly equal in value for the determination of s that we may add them for the final equation, and obtain

$$114{\cdot}65 \times s + 0{\cdot}312 \times M = 35{\cdot}655 \ldots\ldots\ldots\ldots\ldots\ldots (\sigma$$

the equation giving a very probable value for s.

$$\begin{array}{llll}\text{Equation } (\mu) \text{ gives} & 30{\cdot}61 \times M + 155 \times s = 856{\cdot}75 & \ldots\ldots\ldots\ldots(\mu)\\ \tfrac{1}{3}\,(\sigma) & \text{,,} \quad 38{\cdot}22 \times s + 0{\cdot}104 \times M = 11{\cdot}885 & \\ \tfrac{1}{50}\,(\sigma) & \text{,,} \quad 2{\cdot}293 \times s + 0{\cdot}006 \times M = 0{\cdot}713 & \end{array}$$

Adding these to σ we have, neglecting $0{\cdot}16 \times s$, etc.,

$$155 \times s + 0{\cdot}42 \times M = 48{\cdot}25.$$

Subtracting this from (μ) $30{\cdot}19 \times M = 808^{\mathrm{m}}{\cdot}5.$

$$\therefore \qquad M = \frac{80850}{3019} = 26{\cdot}78.$$

Substituting this value for M in (σ)

$$114{\cdot}65 \times s = 35{\cdot}655 - 8{\cdot}355 = 27{\cdot}3,$$

$$\therefore \qquad s = \frac{2730}{11465} = 0{\cdot}237.$$

Substituting M by its value in expression (D) it becomes $53{\cdot}6\left(\frac{p}{57{\cdot}3}\right)^3\{1 + 0{\cdot}014 \times S_h\}^{\mathrm{m}}$, half of which must be subtracted from the mean diurnal luni-tidal interval to give the upper transit luni-tidal interval when the moon's declination is north, and added when the moon's declination is south, and *vice versa* for the lower transit interval. Consequently, when the moon's declination is north, the correction to the mean diurnal luni-tidal interval $7^{\mathrm{h}}\ 36^{\mathrm{m}}{\cdot}8$, to give the time of high water following the moon's upper transit, will be expressed by

$$-\left(\frac{p}{57{\cdot}3}\right)^3\{1 + 0{\cdot}014 \times S_h\}\{26{\cdot}8 \times \sin\delta + 43 \text{ vers } \delta\}^{\mathrm{m}},$$

and that following her lower transit by

$$\left(\frac{p}{57{\cdot}3}\right)^3\{1 + 0{\cdot}014 \times S_h\}\{26{\cdot}8 \times \sin\delta - 43 \text{ vers } \delta\}^{\mathrm{m}},$$

and *vice versa* when the moon's declination is south.

In order to calculate and tabulate these expressions for each hour of the moon's meridian passage, each 5° of the moon's declination to 25° inclusive, and each 4′ of the moon's horizontal parallax from 53′·3 to 61′·3 inclusive, we first calculate the values of $26{\cdot}8 \sin\delta$ and $43 \text{ vers } \delta$ as follows:—

log sin 5° - - -	8·940	2 log sin (2° 30′) - -	$\bar{3}{\cdot}297$
log 26·8 - - -	1·428	log 86 - - -	1·934
log 2·3 - - -	0·368	log 0·2 - - -	$\bar{1}{\cdot}213$

S the sum of these $= 2\cdot5$ and D their difference $= 2\cdot1$.

In like manner for $\delta = 10°$ $S = 5\cdot4$ and $D = 4\cdot0$
$\delta = 15$ $S = 8\cdot4$ „ $D = 5\cdot4$
$\delta = 20$ $S = 11\cdot8$ „ $D = 6\cdot6$
$\delta = 25$ $S = 15\cdot3$ „ $D = 7\cdot3$

log 53·3 - - -	1·727	log 61·3 - - -	1·787
log 57·3 - - -	1·758	log 57·3 - - -	1·758
	$\bar{1}\cdot969$		0·029
	3		3
$\log\left(\frac{53\cdot3}{57\cdot3}\right)^3$ -	$=\bar{1}\cdot907$	$\log\left(\frac{61\cdot3}{57\cdot3}\right)^3$ - - -	0·087

The value of $1 + 0\cdot014\ S_h$ for moon's meridian passage 0^h or 12^h taken from Table IV. is 1·11.

To calculate the upper and lower transit corrections for this, corresponding to moon's declination 5° and moon's horizontal parallax 53·3, we have

log 1·11	0·045	log S	0·398	log D	0·322
log ( )^3	$\bar{1}\cdot907$	log other fact.	$\bar{1}\cdot952$	log other fact.	$\bar{1}\cdot952$
	$\bar{1}\cdot952$	log up. tran. cor.	0·350	log low. tran. cor.	0·274

Upper transit cor. $-2^m\cdot2$. Lower transit cor. $+1^m\cdot9$.

For moon's horizontal parallax 57′·3 and declination 5° we have

log S - - -	0·398	log D - - -	0·322
log 1·11 - - -	0·045	log 1·11 - - -	0·045
log upper tran. cor. -	0·443	log lower tran. cor.	$-0\cdot367$

Upper transit cor. $-2^m\cdot8$. Lower transit cor. $+2^m\cdot3$.

For moon's horizontal parallax 61′·3 and declination 5° for the same hour we have

log S - - -	0·398	log D - - -	0·322
$\log 1\cdot11 + \log\left(\frac{61\cdot3}{57\cdot3}\right)^3$ -	0·132	- - - - -	0·132
log upper tran. cor. -	0·530	log lower tran. cor. -	0·454

Upper transit cor. $-3^m\cdot4$. Lower transit cor. $+2^m\cdot8$.

In this manner the corrections in the following Table (VI.) were calculated.

TABLE VI.

MOON'S DECLINATION CORRECTION TO THE LUNI-TIDAL INTERVAL $7^h\ 36^m{\cdot}8$.

Moon's Meridian Passage.	North Declination, Upper Transit. South ,, Lower ,,			South Declination, Upper Transit. North ,, Lower ,,		
	Horizontal Parallax.			Horizontal Parallax.		
	53′·3	57′·3	61′·3	53′·3	57′·3	61′·3
Declination 5°.						
h	m	m	m	m	m	m
0	− 2·2	− 2·8	− 3·4	+ 1·9	+ 2·3	+ 2·8
1	2·0	2·4	3·0	1·6	2·0	2·5
2	1·7	2·1	2·6	1·4	1·8	2·2
3	1·5	1·8	2·2	1·3	1·6	1·9
4	1·4	1·7	2·1	1·1	1·4	1·7
5	1·4	1·7	2·1	1·1	1·4	1·7
6	1·6	2·0	2·4	1·3	1·6	2·0
7	2·0	2·4	3·0	1·6	2·0	2·5
8	2·5	3·1	3·8	2·1	2·6	3·1
9	2·8	3·4	4·2	2·3	2·9	3·5
10	2·7	3·4	4·1	2·3	2·8	3·5
11	2·5	3·1	3·8	2·1	2·6	3·1
Declination 10°.						
h	m	m	m	m	m	m
0	− 4·8	− 6·0	− 7·3	+ 3·6	+ 4·4	+ 5·4
1	4·2	5·2	6·4	3·1	3·9	4·7
2	3·7	4·6	5·6	2·7	3·4	4·1
3	3·2	4·0	4·9	2·4	3·0	3·6
4	2·9	3·6	4·5	2·2	2·7	3·3
5	2·9	3·6	4·5	2·2	2·7	3·3
6	3·4	4·2	5·1	2·5	3·1	3·8
7	4·2	5·2	6·4	3·1	3·9	4·7
8	5·4	6·7	8·2	4·0	5·0	6·1
9	5·8	7·5	9·1	4·4	5·5	6·7
10	5·7	7·3	8·9	4·3	5·4	6·6
11	5·4	6·7	8·2	4·0	5·0	6·1
Declination 15°.						
h	m	m	m	m	m	m
0	− 7·5	− 9·3	− 11·4	+ 4·8	+ 6·0	+ 7·3
1	6·6	8·2	10·0	4·2	5·2	6·4
2	5·8	7·1	8·7	3·7	4·6	5·6
3	5·0	6·2	7·6	3·2	4·0	4·9
4	4·6	5·7	6·9	2·9	3·6	4·5
5	4·6	5·7	6·9	2·9	3·6	4·5
6	5·3	6·5	8·0	3·4	4·2	5·1
7	6·6	8·2	10·0	4·2	5·2	6·4
8	8·4	10·4	12·7	5·4	6·7	8·2
9	9·4	11·6	14·2	5·8	7·5	9·1
10	9·2	11·3	13·8	5·7	7·3	8·9
11	8·4	10·4	12·7	5·4	6·7	8·2

TABLE VI.—*continued.*

Moon's Meridian Passage.	Declination 25°.					
	North Declination, Upper Transit. South ,, Lower ,,			South Declination, Upper Transit. North ,, Lower ,,		
	Horizontal Parallax.			Horizontal Parallax.		
	53′·3	57′·3	61′·3	53′·3	57′·3	61′·3
	m	m	m	m	m	m
0	−10·6	−13·1	−16·0	+ 5·9	+ 7·3	+ 9·0
1	9·3	11·5	14·0	5·2	6·4	7·8
2	8·1	10·0	12·3	4·5	5·6	6·9
3	7·0	8·7	10·7	3·9	4·9	6·0
4	6·4	8·0	9·7	3·6	4·5	5·4
5	6·4	8·0	9·7	3·6	4·5	5·4
6	7·4	9·2	11·3	4·2	5·2	6·3
7	9·3	11·5	14·0	5·2	6·4	7·8
8	11·8	14·6	17·9	6·6	8·2	10·0
9	13·2	16·3	19·0	7·4	9·2	11·1
10	12·9	15·9	19·5	7·2	8·9	10·9
11	11·8	14·6	17·9	6·6	8·2	10·0
	Declination 20°.					
h	m	m	m	m	m	m
0	−13·7	17·0	−20·7	+ 6·5	+ 8·1	+ 9·9
1	12·0	14·9	18·2	5·7	7·1	8·7
2	10·5	13·0	15·9	5·0	6·2	7·6
3	9·1	11·3	13·8	4·4	5·4	6·6
4	8·3	10·3	12·6	4·0	4·9	6·0
5	8·3	10·3	12·6	4·0	4·9	6·0
6	9·6	11·9	14·6	4·6	5·7	6·9
7	12·0	14·9	18·2	5·7	7·1	8·7
8	15·3	19·0	23·2	7·3	9·0	11·0
9	17·1	21·1	25·8	8·1	10·1	12·3
10	16·7	20·7	25·2	7·9	9·9	12·0
11	15·3	19·0	23·2	7·3	9·0	11·0

Since $s = 0^{m}{\cdot}237$ when the sun's declination was 23°, its value, when the sun's declination is δ', $\dfrac{0^{m}{\cdot}237}{\sin 23^{\circ}} \sin \delta' = 0^{m}{\cdot}61 \sin \delta'$.

Referring to Table V., we see that when the moon's meridian passage is 2^{h} the sun's declination correction is $16 \times 0^{m}{\cdot}66 \sin \delta'$, to be subtracted when the moon's meridian passage is P.M., and the sun's declination north, and to be added for her A.M. transit, and *vice versa* when the sun's declination is south.

To calculate the correction for each 5° of the sun's declination to 23° inclusive, we have

log 11, - - -	1·041
log 0·61, - -	$\bar{1}$·783
log sin 5°, - -	8·940
	$\bar{1}$·764 log correction, $0^m·6$.

In this manner the following table was calculated.

TABLE VII.

THE SUN'S DECLINATION CORRECTION TO LUNI-TIDAL INTERVAL.

Moon's Meridian Passage.	Sun's Declination.						Remarks.
	δ'	5°	10°	15°	20°	23°	
h		m	m	m	m	m	Sun's declination north, add when meridian passage is A.M., and subtract when meridian passage is P.M., and contrariwise when sun's declination is south.
2	11 × 0·61 sin δ'	0·6	1·2	1·7	2·3	2·6	
3	18·5 × 0·61 sin δ'	1·0	2·0	2·9	3·8	4·4	
4	23·5 × 0·61 sin δ'	1·2	2·5	3·7	4·9	5·6	
5	23 × 0·61 sin δ'	1·2	2·4	3·6	4·8	5·5	
6	16 × 0·61 sin δ'	0·8	1·7	2·5	3·3	3·8	

We will now proceed to show how the observations in Table I. are used to construct tables from which the range of the tide and the heights of high and low water respectively may be found sufficiently near for the general purposes of navigation.

Draw through the middle of a foolscap sheet of blue lined paper a straight line AB (Fig. 58) perpendicular to the blue lines. Let it represent the 3 ft. 9 in. mark of the tide pole, which is the average reading of the mean tide line. Take the first blue straight line on the left hand side of the sheet to represent the time of noon on the first of June, and the other blue straight lines in succession from left to right to represent noon on the 2nd, 3rd, 4th, etc., of June respectively; between and parallel to these noon lines draw the red straight lines corresponding to the respective times of the moon's meridian passages on each day.

Referring to Table I., we see that the lower transit high water on the 1st June happened at $5^h\ 17^m$ A.M., and that its height was 5 ft. 9 in.; observing that $5^h\ 17^m$ A.M. is $6^h\ 43^m$ before noon, set the stop of a pair of proportional compasses between the numbers 4 and 3 for dividing lines, at about one-third of the distance between the two lines screw it tight, open the legs, and with the long legs take off along AB the distance between two consecutive blue noon lines; this representing 24^h on the

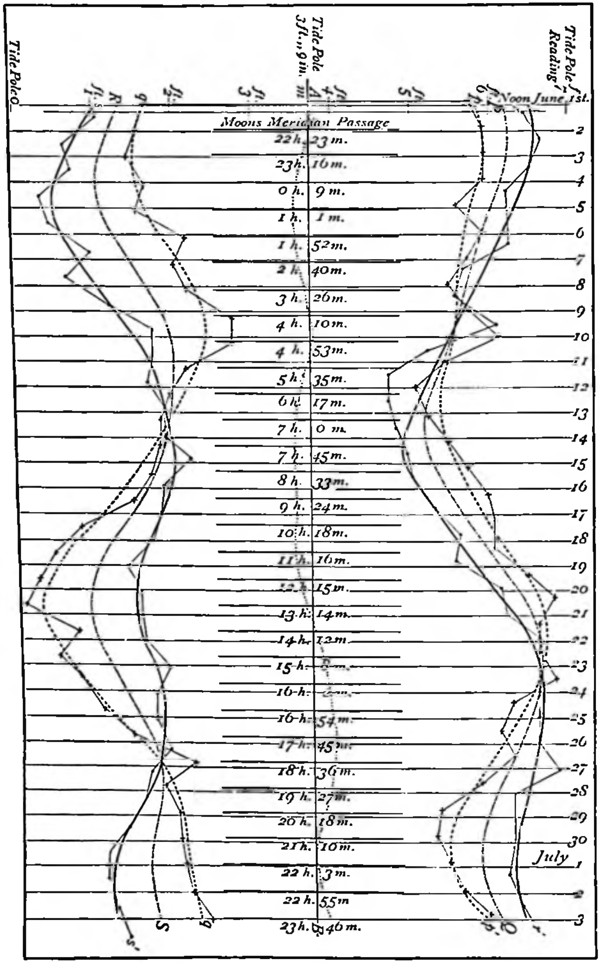

Tide Pole O
Tide Pole 3 ft. 9 in.
Tide Pole ft. Reading
Noon June 1st.
Moons Meridian Passage
22 h. 23 m.
23 h. 16 m.
0 h. 9 m.
1 h. 1 m.
1 h. 52 m.
2 h. 40 m.
3 h. 26 m.
4 h. 10 m.
4 h. 53 m.
5 h. 35 m.
6 h. 17 m.
7 h. 0 m.
7 h. 45 m.
8 h. 33 m.
9 h. 24 m.
10 h. 18 m.
11 h. 16 m.
13 h. 14 m.
14 h.
15 h.
16 h.
16 h. 54 m.
17 h. 45 m.
18 h. 36 m.
19 h. 27 m.
20 h. 18 m.
21 h. 10 m.
22 h. 3 m.
22 h. 55 m.
23 h. 46 m.
July

Fig. 58.

the time scale, the distance between the points of the short legs will represent $6^h\ 43^m$ sufficiently near for our purpose; place the point of the right hand short leg of the compasses on the point where the 1st June noon line intersects *AB*, describe a small circular arc with the point of the left leg cutting *AB* produced to the left in *m*, and through *m* draw the pencil straight line *mp* parallel to the blue straight lines, then all the points in this straight line will correspond to the time $5^h\ 17^m$ A.M. 1st June or very nearly so. From 5 ft. 9 in., the height of the high water, subtract 3 ft. 9 in. the height of the mean tide line represented by *AB*; the difference 2 feet is the height of the high water above the mean tide line, using the inch scale of equal parts which is subdivided to $\frac{1}{12}$ths; take 2 inches from it with a pair of compasses, place one leg of the compasses on the point *m* and with the point of the other leg describe a small circular arc cutting *mp* in *p*, this will denote that the high water on the 1st June happened about $5\frac{1}{4}$ hours A.M., and reached the height of 5 ft. 9 in. This being the high water following the moon's lower transit an ink cross is made with the point *p* at its intersection in order to distinguish the high waters following the moon's lower transits from those following her upper transits, which are denoted by ink dots.

Referring again to Table I. we see that the low water following the high water just projected happened at $11^h\ 17^m$ A.M. on 1st June, and its height was 1 ft. 8 in. At a distance equal to $\frac{1}{32}$ part of that between two consecutive blue lines measured along *AB* and to the left of the 1st June noon line, draw a pencil straight line from *AB* produced to the left downwards parallel to the blue straight lines; this will represent the time of low water sufficiently near for our purpose. Subtract 1ft. 8in. from 3 ft. 9 in., the difference 2 ft. 1 in. shows that the surface at low water was that distance below the mean tide line. With a pair of compasses take $2\frac{1}{12}$ inches from the scale, lay it off on the pencil straight line just drawn from *AB* downwards, and through the point *q* thus determined draw an ink cross to show that it is a lower transit low water. In a similar manner all the intersections of the crosses above *AB* were determined from the times and heights of the lower transit high waters, taken in succession from Table I. Join the points of intersection of the crosses as they follow each other in succession from left to right by pencil straight lines, these would be the chords of the curves representing respectively the high and low waters of the lower transit waves, had they been accurately observed and been undisturbed by wind or other causes, such as the small waves before mentioned which sometimes visit the coast of Nova Scotia. To eliminate, as far as

possible, these abrupt and irregular disturbances we observe that the disturbing forces of the sun and moon change gradually and therefore the curve showing their effects must be of continuous curvature; we therefore draw curved lines of continued curvature through, or as near as possible to, the intersections of the crosses, and so that the area contained between the curve and the straight line AB may slightly exceed the area contained by the straight lines joining the intersections of the crosses and AB respectively. In this manner the dotted curve pp' was drawn to represent the high waters of the lower transit waves, and the dotted curve qq' to represent their low waters.

The black dots above AB are the projections of the high waters of the upper transit waves taken in succession from Table I., those below AB being the projections of their low waters. The black curve rr' above AB shows the most probable heights of the undisturbed upper transit waves derivable from the observations, whilst the black ink curve ss' below AB shows the most probable heights of the low waters of the same waves.

The curve PQ drawn exactly half way between the high water curves rr' and pp' represents the mean diurnal heights of the high waters, and the curve RS drawn exactly half way between the two low water curves qq' and ss' represents the mean diurnal height of the low waters.

We may observe that from the heights given by the RS and PQ curves, the effects of the parts of the disturbing forces involving the odd powers of the sines of the declinations of the sun and moon respectively are eliminated.

The curved dotted line running across the middle of the paper not far from AB, on either side of it, is the mean tide line given by the high water curve PQ and the low water curve RS, between which it is drawn so as to be equidistant from them.

The fluctuations of this curve are, as we have elsewhere described, due to those of the parts of the disturbing forces resolved parallel to the meridian, which remain uncompensated for after one revolution of the earth, combined with the effects of the wind and other disturbances not eliminated from the RS and PQ curves, sufficient of which probably remain to greatly affect the fluctuations of the mean tide curve, dotted, and to make it difficult, if not impossible, to trace the exact relation between the fluctuations of the mean tide line due to those of the solar and lunar elements; however, since the distance between the mean tide curve and the straight line AB is always small, its average value being under 2 inches and its

greatest not more than 5, the small errors introduced by assuming 3 ft. 9 in., as the height of mean tide will be of no practical importance.

We will now show how the values of the ranges of the tide given by the curves are used to construct tables from which future ranges can be predicted with sufficient accuracy for the general purposes of navigation.

Let R be the range of the lunar tide expressed in feet; when the moon is on the equator and her horizontal parallax $57'\cdot 3$, let an increase of $1'$ in the moon's horizontal parallax cause an increase of m feet in the range of her tide, and when the moon's declination is δ let the consequent increase in the mean diurnal lunar range be n vers δ feet; consequently when the moon's horizontal parallax is $57'\cdot 3+\delta p$ and her declination δ, the mean diurnal lunar range will be expressed by $R+m\delta p+n$ vers δ.

We will now proceed to express symbolically the part of the range of the combined wave which is due to the solar wave.

Let the mean diurnal range of the solar wave during the period of observation contained in Table I. be $2s$, its high water rising s feet above the mean tide line and its low water falling s feet below that line; at one hour before or after the time of the solar high water let the height of the solar wave above the mean tide line be as feet; at one hour before and after the time of the solar low water will consequently be as feet below the mean tide line; at two hours before and after the time of the solar high water let the height of the solar wave be bs feet above the mean tide line, and therefore at two hours before and after the solar low water the height of the solar wave will be bs feet below the mean tide line. At three hours before and after the solar high water, which will coincide with three hours after or before the solar low water, the solar wave will be at the mean tide level.

We have seen that when the moon passes the meridian at 1^h or 13^h the lunar and solar high waters happen at the same time, or nearly so, and in this case the range of the combined wave will be sum of the lunar and solar ranges, and the mean diurnal range will therefore be expressed by

$$R+m\delta p+n \text{ vers } \delta+2s \text{......................(1)}$$

When the moon passes the meridian at 2^h, 12^h, 14^h, or 24^h the mean diurnal range will be given by

$$R+m\delta p+n \text{ vers } \delta+2as \text{......................(2)}$$

When the moon passes the meridian at 3^h, 11^h, 15^h, or 23^h the mean diurnal range of the combined wave

$$=R+m\delta p+n \text{ vers } \delta+2bs \text{...................(3)}$$

When the moon passes the meridian at 4^h, 10^h, 16^h, or 22^h the mean diurnal range of the tide will be expressed by

$$R + m\delta p + n \text{ vers } \delta \quad \text{.......................(4)}$$

When the moon passes the meridian at 5^h, 9^h, 17^h, or 21^h the mean diurnal range will be given by

$$R + m\delta p + n \text{ vers } \delta - 2bs \quad \text{.....................(5)}$$

When the moon passes the meridian at 6^h, 8^h, 18^h, or 20^h the mean diurnal range will be

$$= R + m\delta p + n \text{ vers } \delta - 2as \quad \text{...................(6)}$$

Lastly, when the moon passes the meridian at 7^h or 19^h we shall have for the mean diurnal range

$$R + m\delta p + n \text{ vers } \delta - 2s \quad \text{......................(7)}$$

Referring to Fig. 58, through the point r representing the high water following the moon's upper transit on the 31st May draw a straight pencil line parallel to the noon lines cutting all the curved lines both above and below AB, the part of this line intercepted between the two PQ and RS curves will give the mean diurnal range for moon's meridian passage 21^h 30^m, which, when measured, will be found to be $4\frac{10\frac{1}{2}}{12}$ ins., this on our scale represents 4 ft. $10\frac{1}{2}$ in. Next measure the part of the straight pencil line above AB which is intercepted between the PQ and rr' curved lines, this will be found $=\frac{4}{12}$ ins., which represents 4 ins., and shows that the upper transit high water was 4 ins. above the mean diurnal height of high water. Next measure the length of the pencil straight line intercepted between the RS and qq' curved lines below AB, this will be found $=\frac{3\frac{1}{2}}{12}$ ins., and shows that the upper transit low water was $3\frac{1}{2}$ ins. below the mean diurnal low water: consequently the upper transit range exceeded the mean diurnal range by $7\frac{1}{2}$ ins. Proceeding in this manner by drawing straight pencil lines parallel to the noon lines through all the dots above AB we obtain the quantities inserted in the following Table VIII., remarking that the moon's horizontal parallax and declination inserted in the sixth and seventh columns correspond to the time of the moon's meridian passage inserted in the first.

Table VIII.

MEAN DIURNAL RANGE GIVEN BY *PQ* AND *RS* CURVES, AND DIURNAL INEQUALITY OF RANGE.

Moon's Meridian Passage.	Mean Diurnal Range.	Moon's Upper Transit. High Water above Mean.	Moon's Upper Transit. Low Water below Mean.	Moon's Upper Transit. Range greater than Mean.	Moon's Horizontal Parallax.	Moon's Declination
h. m.	ft. in.	in.	in.	in.	′	°
21 30	4 10½	4	3½	7½	57·9	14·5 N.
22 23	5 1	4	4½	8½	57·4	17·5
23 16	5 2	3¼	5	8¼	56·9	19·5
0 9	5 1	3½	6	9½	56·3	20·3
1 1	4 10¼	3	7	10	55·7	20·0
1 52	4 7	2¾	7¼	10	55·2	18·6
2 40	4 3½	2	7½	9½	54·8	16·3
3 26	3 11½	1	7	8	54·4	13·3
4 10	3 8½	½	6	6½	54·2	9·8
4 53	3 5¼	− ¼	4¾	4½	54·2	5·9
5 35	3 4½	1	3	2	54·3	1·7 N.
6 17	3 3½	2	1¼	− ¾	54·6	2·5 S.
7 0	3 3¼	3	− ¼	3¼	55·1	6·7
7 45	3 5	4	2¼	6¼	55·8	10·8
8 33	3 8½	4	3¼	7¼	56·5	14·8
9 24	4 1	4	4¼	8¼	57·4	17·3
10 18	4 6½	3½	5¼	8¾	58·2	19·4
11 16	4 11¼	3½	6¼	9¾	58·9	20·3
12 15	5 4	3¼	7	10¼	59·6	19·9
13 14	5 7½	2½	7½	10	60·0	17·9
14 12	5 8	1	7	8	60·2	14·8
15 8	5 6	0	6¼	6¼	60·1	10·6
16 2	5 3¼	+ ¾	5¼	4½	59·9	5·7
16 54	5 0¾	1¾	3¾	2	59·5	0·6 S.
17 45	4 9	2¾	1¾	+ 1	58·9	4·6 N
18 36	4 6	4	+ ¼	4¼	58·4	9·1
19 27	4 3¾	5	2	7	57·8	13·3
20 18	4 3	5	3¾	8¾	57·3	16·5
21 10	4 2¾	5¼	5¼	10½	56·7	18·9
22 3	4 3¼	5	6	11	56·2	20·1
22 55	4 4	4	6½	10½	55·7	20·2
23 46	4 6	2½	7	9½	55·2	19·3
0 35	4 6¼	1¾	7¾	9½	54·6	17·3
1 22	4 5½	1	7¾	8¾	54·4	14·6
2 7	4 3½	¼	7¼	7½	54·2	11·3
2 50	4 1	0	7	7	54·1	7·5
3 32	3 10¼	− ¼	6	5¾	54·1	3·4
4 13	3 8	½	5	4½	54·2	0·7 S.
4 55	3 6¼	1	3½	2½	54·5	4·9
5 39	3 5¼	2	1½	− ½	55·1	9·0
6 24	3 4¾	3	− 1	− 4	55·7	12·7
7 12	3 4¾	4	2½	6½	56·5	16·0
8 4	3 6	5	3½	8½	57·4	18·5
9 0	3 9	4¾	4	8¾	58·4	20·0

The following Table IX. was constructed from Table VIII. in order to give the equations more conveniently for elimination.

Table IX.

MEAN DIURNAL RANGE OF TIDE CALCULATED FROM TABLE VIII.

Moon's Meridian Passage.	Mean Diurnal Range.	Moon's Upper Transit.			Moon's	
		High Water above Mean.	Low Water below Mean	Range greater than Mean.	Horizontal Parallax.	Declination
h.	ft. in.	in.	in.	in.	°	°
22	5 0	3⅞	4¼	8⅛	57·6	16·0 N.
23	5 1·7	3⅞	4¾	8⅝	57·1	19·0
0	5 1¼	3½	5¾	9¼	56·4	20·0
1	4 10¼	3	7	10	55·7	20·0
2	4 6¼	2⅝	7¼	9⅞	55·1	18·0
3	4 1⅜	1½	7¼	8¾	54·6	15·0
4	3 9¼	⅜	6¼	6⅝	54·3	11·0
5	3 5½	− ⅜	4½	4⅛	54·2	5·0
6	3 3⅞	1¾	1¾	0	54·5	1·0 S.
7	3 3¼	3	− ¾	−3¾	55·1	6·7
8	3 6	4	2¾	6¾	56·0	12·0 S.
9	3 10⅞	4	3¾	7¾	57·0	16·0
10	4 4⅝	3⅝	5	8⅝	58·0	18·7
11	4 10	3½	5⅞	9⅜	58·7	20·0
12	5 2¾	3¼	6⅞	10⅛	59·4	20·0
13	5 6½	2⅜	7⅝	10	59·9	18·4
14	5 8	1⅛	7	8⅛	60·2	15·5
15	5 6¼	⅛	6⅜	6½	60·1	11·0
16	5 3¼	+ ¾	5¼	4½	59·9	5·7
17	5 0¼	1⅞	3½	1⅝	59·4	0·3
18	4 8	3⅛	1⅛	+2	58·7	5·4 N.
19	4 5	4⅜	+1	5⅜	58·1	10·8
20	4 3¼	5	3⅛	8⅛	57·5	15·4
21	4 2¾	5¼	4⅞	10⅛	56·8	18·4
22	4 3¼	5	6	11	56·2	20·1
23	4 4½	3⅞	6½	10⅜	55·6	20·1
0	4 6	2¼	7¼	9½	55·0	18·5
1	4 5⅞	1⅝	7½	9⅛	54·5	15·9
2	4 3¾	⅜	7¼	7⅝	54·2	11·8
3	4 0⅜	0	6⅞	6⅞	54·1	6·5
4	3 8⅞	− ⅜	5¼	4⅞	54·2	0·4
5	3 6⅛	1⅛	3¼	1⅝	54·6	5·4 S.
6	3 5	2½	⅜	−2⅛	55·3	10·8 S.
7	3 4¾	3¾	−2½	5⅝	56·3	15·2
8	3 6	5	3½	8½	57·4	18·5
9	3 9	4¾	4	8¾	58·4	20·0

Equation (4) gives the symbolical expression for the mean diurnal range when the moon passes the meridian at 22^h

taking the values from the first line in Table IX. we have mean diurnal range 5 ft. 0 in.; $\delta p = 0'{\cdot}3$ and $\delta = 16°$; taking the natural versine of this last to three places of decimals, and substituting the symbols by their values, we have

$$R + 0{\cdot}3 \times m + 0{\cdot}039 \times n \qquad = 5 \text{ ft. } 0 \text{ in.} \quad \text{(I)}$$

In like manner, for meridian passage 23, equation (3) gives

$$R - 0{\cdot}2 \times m + 0{\cdot}054 \times n + 2bs = 5 \text{ ft. } 1{\cdot}7 \text{ in.} \quad \text{(II)}$$

The third line and equation (2) give

$$R - 0{\cdot}9 \times m + 0{\cdot}060 \times n + 2as = 5 \text{ ft. } 1{\cdot}2 \text{ in.} \quad \text{(III)}$$

Proceeding in this way with the fourth and each succeeding line on Table IX. we find

$$\begin{array}{lrrl}
R - 1{\cdot}6 \times m + 0{\cdot}060 \times n + 2s & = 4 \text{ ft.} & 10{\cdot}2 \text{ in.} & \text{(IV)} \\
R - 2{\cdot}2 \times m + 0{\cdot}049 \times n + 2as & = 4 & 6{\cdot}5 & \text{(V)} \\
R - 2{\cdot}7 \times m + 0{\cdot}034 \times n + 2bs & = 4 & 1{\cdot}6 & \text{(VI)} \\
R - 3{\cdot}0 \times m + 0{\cdot}018 \times n & = 3 & 9{\cdot}2 & \text{(VII)} \\
R - 3{\cdot}1 \times m + 0{\cdot}004 \times n - 2bs & = 3 & 5{\cdot}5 & \text{(VIII)} \\
R - 2{\cdot}8 \times m \qquad\qquad - 2as & = 3 & 3{\cdot}9 & \text{(IX)} \\
R - 2{\cdot}2 \times m + 0{\cdot}007 \times n - 2s & = 3 & 2{\cdot}2 & \text{(X)} \\
R - 1{\cdot}3 \times m + 0{\cdot}022 \times n - 2as & = 3 & 6 & \text{(XI)} \\
R - 0{\cdot}3 \times m + 0{\cdot}039 \times n - 2bs & = 3 & 10{\cdot}9 & \text{(XII)} \\
R + 0{\cdot}7 \times m + 0{\cdot}053 \times n - & = 4 & 4{\cdot}6 & \text{(XIII)} \\
R + 1{\cdot}4 \times m + 0{\cdot}060 \times n + 2bs & = 4 & 10 & \text{(XIV)} \\
R + 2{\cdot}1 \times m + 0{\cdot}060 \times n + 2as & = 5 & 2{\cdot}7 & \text{(XV)} \\
R + 2{\cdot}6 \times m + 0{\cdot}051 \times n + 2s & = 5 & 6{\cdot}5 & \text{(XVI)} \\
R + 2{\cdot}9 \times m + 0{\cdot}036 \times n + 2as & = 5 & 8 & \text{(XVII)} \\
R + 2{\cdot}8 \times m + 0{\cdot}018 \times n + 2bs & = 5 & 6{\cdot}2 & \text{(XVIII)} \\
R + 2{\cdot}6 \times m + 0{\cdot}005 \times n & = 5 & 3{\cdot}2 & \text{(XIX)} \\
R + 2{\cdot}1 \times m \qquad\qquad - 2bs & = 5 & 0{\cdot}2 & \text{(XX)} \\
R + 1{\cdot}4 \times m + 0{\cdot}004 \times n - 2as & = 4 & 8 & \text{(XXI)} \\
R + 0{\cdot}8 \times m + 0{\cdot}017 \times n - 2s & = 4 & 5 & \text{(XXII)} \\
R + 0{\cdot}2 \times m + 0{\cdot}035 \times n - 2as & = 4 & 3{\cdot}2 & \text{(XXIII)} \\
R - 0{\cdot}5 \times m + 0{\cdot}051 \times n - 2bs & = 4 & 2{\cdot}7 & \text{(XXIV)} \\
R - 1{\cdot}1 \times m + 0{\cdot}061 \times n & = 4 & 3{\cdot}2 & \text{(XXV)} \\
R - 1{\cdot}7 \times m + 0{\cdot}061 \times n + 2bs & = 4 & 4{\cdot}2 & \text{(XXVI)} \\
R - 2{\cdot}3 \times m + 0{\cdot}052 \times n + 2as & = 4 & 6 & \text{(XXVII)} \\
R - 2{\cdot}8 \times m + 0{\cdot}038 \times n + 2s & = 4 & 5{\cdot}9 & \text{(XXVIII)} \\
R - 3{\cdot}1 \times m + 0{\cdot}021 \times n + 2as & = 4 & 3{\cdot}8 & \text{(XXIX)} \\
R - 3{\cdot}2 \times m + 0{\cdot}006 \times n + 2bs & = 4 & 0{\cdot}3 & \text{(XXX)}
\end{array}$$

$$R-3{\cdot}1\times m+ \qquad =3 \text{ ft. } 8{\cdot}8 \text{ in.} \quad \text{(XXXI)}$$
$$R-2{\cdot}7\times m+0{\cdot}004\times n-2bs=3 \quad 6{\cdot}1 \quad \text{(XXXII)}$$
$$R-2{\cdot}0\times m+0{\cdot}018\times n-2as=3 \quad 5 \quad \text{(XXXIII)}$$
$$R-1{\cdot}0\times m+0{\cdot}035\times n-2s=3 \quad 4{\cdot}8 \quad \text{(XXXIV)}$$
$$R+0{\cdot}1\times m+0{\cdot}052\times n-2as=3 \quad 6 \quad \text{(XXXV)}$$
$$R+1{\cdot}1\times m+0{\cdot}060\times n-2bs=3 \quad 9 \quad \text{(XXXVI)}$$

To eliminate s add (I) and (VII), (II) and (VII), ... (XXX) and (XXXVI), which give the following eighteen equally valuable equations, none of which contain s.

$$2R-2{\cdot}7\times m+0{\cdot}057\times n=8 \text{ ft. } 9{\cdot}2 \text{ in.} \quad (1)$$
$$2R-3{\cdot}3\times m+0{\cdot}058\times n=8 \quad 7{\cdot}2 \quad (2)$$
$$2R-3{\cdot}7\times m+0{\cdot}060\times n=8 \quad 5{\cdot}1 \quad (3)$$
$$2R-3{\cdot}8\times m+0{\cdot}067\times n=8 \quad 1{\cdot}4 \quad (4)$$
$$2R-3{\cdot}5\times m+0{\cdot}071\times n=8 \quad 0{\cdot}5 \quad (5)$$
$$2R-3{\cdot}0\times m+0{\cdot}073\times n=8 \quad 0{\cdot}5 \quad (6)$$
$$2R+3{\cdot}3\times m+0{\cdot}058\times n=9 \quad 7{\cdot}8 \quad (7)$$
$$2R+3{\cdot}5\times m+0{\cdot}060\times n=9 \quad 10{\cdot}2 \quad (8)$$
$$2R+3{\cdot}5\times m+0{\cdot}064\times n=9 \quad 10{\cdot}7 \quad (9)$$
$$2R+3{\cdot}4\times m+0{\cdot}068\times n=9 \quad 11{\cdot}5 \quad (10)$$
$$2R+3{\cdot}1\times m+0{\cdot}071\times n=9 \quad 11{\cdot}2 \quad (11)$$
$$2R+2{\cdot}3\times m+0{\cdot}069\times n=9 \quad 8{\cdot}9 \quad (12)$$
$$2R-4{\cdot}2\times m+0{\cdot}061\times n=8 \quad 0 \quad (13)$$
$$2R-4{\cdot}4\times m+0{\cdot}065\times n=7 \quad 10{\cdot}3 \quad (14)$$
$$2R-4{\cdot}3\times m+0{\cdot}070\times n=7 \quad 11 \quad (15)$$
$$2R-3{\cdot}8\times m+0{\cdot}073\times n=7 \quad 10{\cdot}7 \quad (16)$$
$$2R-3{\cdot}0\times m+0{\cdot}073\times n=7 \quad 9{\cdot}8 \quad (17)$$
$$2R-2{\cdot}1\times m+0{\cdot}066\times n=7 \quad 7{\cdot}9 \quad (18)$$

The first six of these equations added together give

$$12R-20{\cdot}0\times m+0{\cdot}386\times n=49 \text{ ft. } 11{\cdot}9 \text{ in.} \quad \text{(A)}$$

The next six added together give

$$12R+19{\cdot}1\times m+0{\cdot}390\times n=59 \text{ ft. } 0{\cdot}3 \text{ in.} \quad \text{(B)}$$

The last six added give

$$12R-21{\cdot}8\times m+0{\cdot}408\times n=47 \text{ ft. } 2{\cdot}8 \text{ in.} \quad \text{(C)}$$

Taking the arithmetic mean of (A) and (C) we have

$$12R-20{\cdot}9\times m+0{\cdot}397\times n=48 \text{ ft. } 7{\cdot}35 \text{ in.} \quad \text{(D)}$$

Subtracting this from (B)

$$40\times m-0{\cdot}007\times n=10 \text{ ft. } 4{\cdot}95 \text{ in.} \quad \text{(E)}$$

Neglecting the term involving n, as practically too small, we have

$$m = \frac{124{\cdot}95 \text{ in.}}{40} = 3{\cdot}124 \text{ in.}$$

Adding (D) and (B),

$$24R - 1{\cdot}8 \times m + 0{\cdot}787 \times n = 107 \text{ ft. } 7{\cdot}65 \text{ in.} \ldots\ldots\ldots\ldots (\text{G})$$

Substituting m by its value and transposing,

$$24R + 0{\cdot}787 \times n = 107 \text{ ft. } 7{\cdot}65 \text{ in.} + 5{\cdot}62 \text{ in} = 108 \text{ ft. } 1{\cdot}3 \text{ in.} \ldots (\text{K})$$

Adding (XIII) and (XXV) we have

$$2R - 0{\cdot}4 \times m + 0{\cdot}114 \times n = 8 \text{ ft. } 7{\cdot}8 \text{ in.}$$

Adding (XIX) and (XXXI) we have

$$2R - 0{\cdot}5 \times m + 0{\cdot}005 \times n = 9 \text{ ft. } 0 \text{ in.}$$

Subtracting these

$$0{\cdot}1 \times m + 0{\cdot}109 \times n = -4{\cdot}2 \text{ in.}$$

Replacing m by its value and transposing,

$$0{\cdot}109 \times n = -4{\cdot}2 \text{ in.} - 0{\cdot}3 \text{ in.} = -4{\cdot}5 \text{ in.} \ldots\ldots\ldots\ldots (\text{L})$$

Adding (XIV) and (XVI) we have

$$2R - 0{\cdot}3 \times m + 0{\cdot}121 \times n + 4bs = 9 \text{ ft. } 2{\cdot}2 \text{ in.}$$

Adding (XVIII) and (XXX) we have

$$2R - 0{\cdot}4 \times m + 0{\cdot}024 \times n + 4bs = 9 \text{ ft. } 6{\cdot}5 \text{ in.}$$

Subtracting

$$0{\cdot}1 \times m + 0{\cdot}097 \times n = -4{\cdot}3 \text{ in.}$$

Replacing m by its value and transposing,

$$0{\cdot}097 \times n = -4{\cdot}3 \text{ in.} - 0{\cdot}3 \text{ in.} = -4{\cdot}6 \text{ in.} \ldots\ldots\ldots\ldots (\text{M})$$

Adding (III) and (XV) we have

$$2R + 1{\cdot}2 \times m + 0{\cdot}120 \times n + 4as = 10 \text{ ft. } 3{\cdot}9 \text{ in.}$$

Adding (XVII) and (XXIX) we have

$$2R - 0{\cdot}2 \times m + 0{\cdot}057 \times n + 4as = 9 \text{ ft. } 11{\cdot}8 \text{ in.}$$

Subtracting, replacing m by its value, and transposing,

$$0{\cdot}063 \times n = +4{\cdot}1 \text{ in.} - 4{\cdot}4 \text{ in.} = -0{\cdot}3 \text{ in.} \ldots\ldots\ldots\ldots (\text{N})$$

Adding (VIII) and (XX) we have

$$2R - m + 0{\cdot}004 \times n - 4bs = 8 \text{ ft. } 5{\cdot}7 \text{ in.}$$

Adding (XXIV) and (XXXVI) we have

$$2R+0{\cdot}6\times m+0{\cdot}111\times n-4bs=7\text{ ft. }11{\cdot}7\text{ in.}$$

Subtracting, replacing m by its value, and we have

$$0{\cdot}107\times n=-6\text{ in.}-5\text{ in.}=-11\text{ in.}\ldots\ldots\ldots\ldots\ldots(\text{P})$$

Adding (IX) and (XXI) we have

$$2R-1{\cdot}4\times m+0{\cdot}004\times n-4as=7\text{ ft. }11{\cdot}9\text{ in.}$$

Adding (XXIII) and (XXXV) we have

$$2R+0{\cdot}3\times m+0{\cdot}087\times n-4as=7\text{ ft. }9{\cdot}2\text{ in.}$$

Subtracting in the same manner as before

$$0{\cdot}083\times n=-2{\cdot}7\text{ in.}-5{\cdot}3\text{ in.}=-8\text{ in.}\ldots\ldots\ldots\ldots(\text{Q})$$

Subtracting (X) from (XXXIV) we have

$$1{\cdot}2\times m+0{\cdot}028\times n=1{\cdot}6\text{ in.}$$

Subtracting (XXVIII) from (IV) we have

$$1{\cdot}2\times m+0{\cdot}022\times n=4{\cdot}3\text{ in.}$$

Adding these, substituting m by its value, and transposing,

$$0{\cdot}050\times n=5{\cdot}9\text{ in.}-7{\cdot}5\text{ in.}=-1{\cdot}6\text{ in.}\ldots\ldots\ldots\ldots\ldots(\text{R})$$

Observing that the equations (L), (M), ... (R) are each derived from four of the original equations in such a manner as to give as large a coefficient to n as possible, we may consider their values as proportional to those coefficients; therefore we multiply (L) by 11, (M) by 10, (N) by 6, (P) by 11, (Q) by 8, and (R) by 5, because these are as nearly in those proportions as small whole numbers can be obtained, and will practically give as good a result as if the large exact numbers were taken.

(L) × 11 gives	$1{\cdot}199\times n=$	$-\ \ 49{\cdot}5$ in.
(M) × 10	$0{\cdot}970\times n=$	$-\ \ 46{\cdot}0$
(N) × 6	$0{\cdot}378\times n=$	$-\ \ \ 1{\cdot}8$
(P) × 11	$1{\cdot}177\times n=$	-121
(Q) × 8	$0{\cdot}664\times n=$	$-\ \ 64$
(R) × 5	$0{\cdot}250\times n=$	$-\ \ \ 8$

Adding, $4{\cdot}638\times n=-290{\cdot}3;$

$$\therefore\qquad n=-\frac{290{\cdot}3}{4{\cdot}638}=-62{\cdot}6\text{ in.}$$

From (K) $24\ R = 108$ ft. $1{\cdot}3$ in. $- \ 0{\cdot}787 \times n$ in.
$= 108 \quad 1{\cdot}3 \quad + 49{\cdot}3$ in.
$= 112 \quad 2{\cdot}6$

$\therefore \qquad R = \ 4$ ft. $8{\cdot}11$ in.

To find the values of s, a, and b

Subtract equation (X) from (IV), $4s + 0{\cdot}6 \times m + 0{\cdot}053 \times n = 19$ in.
(XXII) (XVI), $4s + 1{\cdot}8 \times m + 0{\cdot}034 \times n = 13{\cdot}5$
(XXIV) (XXVIII), $4s - 1{\cdot}8 \times m + 0{\cdot}003 \times n = 13{\cdot}1$

Adding these and transposing,

$$12s = 45{\cdot}6 \text{ in.} - 0{\cdot}6 \times m - 0{\cdot}09 \times n$$
$$= 45{\cdot}6 - 1{\cdot}9 + 5{\cdot}6 = 49{\cdot}3 \text{ in.}$$
$$s = \frac{49{\cdot}3 \text{ in.}}{12} = 4{\cdot}11 \text{ in.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{S})$$

Adding (III) and (V), $2R - 2{\cdot}9 \times m + 0{\cdot}109 \times n + 4as = 9$ ft. $7{\cdot}7$ in.
(IX) (XI), $2R - 4{\cdot}1 \times m + 0{\cdot}022 \times n - 4as = 6 \quad 9{\cdot}9$

Subtracting and transposing,

$$8as = 33{\cdot}8 \text{ in.} - 1{\cdot}2 \times m - 0{\cdot}087 \times n.$$

Adding (XV) and (XVII),

$$2R + 5{\cdot}0 \times m + 0{\cdot}096 \times n + 4as = 10 \text{ ft. } 10{\cdot}7 \text{ in.}$$

Adding (XXI) and (XXIII),

$$2R + 1{\cdot}6 \times m + 0{\cdot}039 \times n - 4as = 8 \text{ ft. } 11{\cdot}2 \text{ in.}$$

Subtracting and transposing terms involving m and n,

$$8as = 23{\cdot}5 - 3{\cdot}4 \times m - 0{\cdot}057 \times 72.$$

Adding (XXVII) and (XXIX),

$$2R - 5{\cdot}4 \times m + 0{\cdot}073 \times n + 4as = 8 \text{ ft. } 9{\cdot}8 \text{ in.}$$

Adding (XXXIII) and (XXXV),

$$2R - 1{\cdot}9 \times m + 0{\cdot}070 \times n - 4as = 6 \text{ ft. } 11 \text{ in.}$$

Subtracting and transposing terms involving m and n,

$$8as = 22{\cdot}8 + 3{\cdot}5 \times m - 0{\cdot}003 \times n.$$

Adding these three values of $8as$ we have

$$24as = 80{\cdot}1 \text{ in.} - 1{\cdot}1 \times m - 0{\cdot}147 \times n$$
$$= 80{\cdot}1 \quad - 3{\cdot}4 + 9{\cdot}2$$
$$= 85{\cdot}9$$

$$\therefore \qquad as = \frac{85{\cdot}9}{24} = 3{\cdot}6 \text{ in.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

Adding (II) and (VI),

$$2R-2{\cdot}9\times m+0{\cdot}088\times n+4bs=9\text{ ft. }3{\cdot}3\text{ in.}$$

Adding (VIII) and (XII),

$$2R-3{\cdot}4\times m+0{\cdot}043\times n-4bs=7\text{ ft. }4{\cdot}4\text{ in.}$$

Subtracting and transposing terms involving m and n,

$$8bs=22{\cdot}9\text{ in.}-0{\cdot}5\times m-0{\cdot}045\times n.$$

Adding (XIV) and (XVIII),

$$2R+4{\cdot}2\times m+0{\cdot}078\times n+4bs=10\text{ ft. }4{\cdot}2\text{ in.}$$

Adding (XX) and (XXIV),

$$2R+1{\cdot}6\times m+0{\cdot}051\times n-4bs=9\text{ ft. }2{\cdot}9\text{ in.}$$

Difference gives $8bs=13{\cdot}3\text{ in.}-2{\cdot}6\times m-0{\cdot}027\times n.$

Adding (XXVI) and (XXX),

$$2R-4{\cdot}9\times m+0{\cdot}067\times n+4bs=8\text{ ft. }4{\cdot}5\text{ in.}$$

Adding (XXXII) and (XXXVI),

$$2R-1{\cdot}6\times m+0{\cdot}064\times n-4bs=7\text{ ft. }3{\cdot}1\text{ in.}$$

Subtracting and transposing terms involving m and n,

$$8bs=13{\cdot}4\text{ in.}+3{\cdot}3\times m-0{\cdot}003\times n.$$

Adding these three values of $8bs$ we have

$$\begin{aligned}24bs&=49{\cdot}6\text{ in.}+0{\cdot}2\times m-0{\cdot}075\times n\\&=49{\cdot}6\text{ in.}+0{\cdot}6\text{ in.}+4{\cdot}7\text{ in.}=54{\cdot}9\end{aligned}$$

$$\therefore\qquad bs=\frac{54{\cdot}9\text{ in.}}{24}=2{\cdot}3\text{ in.}\ldots\ldots\ldots\ldots\ldots\ldots(\beta)$$

Taking the equations in which the term involving n is not large, and introducing in them the values of m, n, s, as and bs, we obtain a new value of R as follows.

Equation (VII) gives	$R=3\text{ ft. }9{\cdot}2\text{ in.}+3{\cdot}0\times m-0{\cdot}018\times n$
(VIII)	$R=3\text{ ft. }5{\cdot}5\text{ in.}+3{\cdot}1\times m-0{\cdot}014\times n+2bs$
(IX)	$R=3\text{ ft. }3{\cdot}9\text{ in.}+2{\cdot}8\times m\qquad+2as$
(X)	$R=3\text{ ft. }3{\cdot}2\text{ in.}+2{\cdot}2\times m-0{\cdot}007\times n+2s$
(XI)	$R=3\text{ ft. }6\text{ in.}+1{\cdot}3\times m-0{\cdot}022\times n+2as$
(XXIX)	$R=4\text{ ft. }3{\cdot}8\text{ in.}+3{\cdot}1\times m-0{\cdot}021\times n-2as$
(XXX)	$R=4\text{ ft. }0{\cdot}3\text{ in.}+3{\cdot}2\times m-0{\cdot}006\times n-2bs$
(XXXI)	$R=3\text{ ft. }8{\cdot}8\text{ in.}+3{\cdot}1\times m$
(XXXII)	$R=3\text{ ft. }6{\cdot}1\text{ in.}+2{\cdot}7\times m-0{\cdot}004\times n+2bs$
(XXXIII)	$R=3\text{ ft. }5\text{ in.}+2{\cdot}0\times m-0{\cdot}018\times n+2as$

Adding, we have

$$10R = 36 \text{ ft. } 3{\cdot}8 \text{ in.} + 26{\cdot}5 \times m - 0{\cdot}110 \times n + 2bs + 4as + 2s$$

Dividing by 2,

$$5R = 18 \text{ ft. } 1{\cdot}9 \text{ in.} \times 13{\cdot}25 \times m - 0{\cdot}055 \times n + bs + 2as + s$$

$$\begin{array}{ll} \text{Equation (XVII) gives} & R = 5 \text{ ft. } 8 \quad \text{in.} - 2{\cdot}9 \times m - 0{\cdot}036 \times n - 2as \\ \text{(XVIII)} & R = 5 \text{ ft. } 6{\cdot}2 \text{ in.} - 2{\cdot}8 \times m - 0{\cdot}018 \times n - 2bs \\ \text{(XIX)} & R = 5 \text{ ft. } 3{\cdot}2 \text{ in.} - 2{\cdot}6 \times m - 0{\cdot}005 \times n \\ \text{(XX)} & R = 5 \text{ ft. } 0{\cdot}2 \text{ in.} - 2{\cdot}1 \times m \qquad\qquad + 2bs \\ \text{(XXI)} & R = 4 \text{ ft. } 8 \quad \text{in.} - 1{\cdot}4 \times m - 0{\cdot}004 \times n + 2as \\ \text{(XXII)} & R = 4 \text{ ft. } 5 \quad \text{in.} - 0{\cdot}8 \times m - 0{\cdot}017 \times n + 2s \end{array}$$

Adding these six equations,

$$6R = 30 \text{ ft. } 6{\cdot}6 \text{ in.} - 12{\cdot}6 \times m - 0{\cdot}080 \times n + 2s$$

But $5R = 18 \text{ ft. } 1{\cdot}9 \text{ in.} + 13{\cdot}25 \times m - 0{\cdot}055 \times n + s + 2as + bs$

Adding, we have

$$11R = 48 \text{ ft. } 8{\cdot}3 \text{ in.} + 0{\cdot}65 \times m - 0{\cdot}135 \times n + 3s + 2as + bs$$

Replacing m, n, s, as, and bs by their values

$$\begin{aligned} 11R &= 48 \text{ ft. } 8{\cdot}3 \text{ in.} + 2{\cdot}0 \text{ in.} + 8{\cdot}4 \text{ in.} + 12{\cdot}3 \text{ in.} + 7{\cdot}2 + 2{\cdot}3 \\ &= 51 \text{ ft. } 4{\cdot}5 \text{ in.} \end{aligned}$$

$$R = \frac{51 \text{ ft. } 4{\cdot}5 \text{ in.}}{11} = 4 \text{ ft. } 8 \text{ in.}$$

We therefore adopt

$$R = 4 \text{ ft. } 8 \text{ in.}; \; m = 3{\cdot}1 \text{ in.}; \; n = -60 \text{ in.}$$

$$s = 4{\cdot}1 \text{ in.}; \; a = \frac{36}{41}; \text{ and } b = \frac{23}{41}$$

as good practical values.

Let the moon's declination expressed in degrees be δ, and her horizontal parallax in minutes of angle p; then $D\left(\frac{p}{57{\cdot}3}\right)^3 \sin\delta$, where D may be considered constant, will express approximately the difference between the height of the high water of the superior transit lunar wave and that of the corresponding mean diurnal lunar wave; it will also express the difference between the heights of their low waters; and therefore the difference between the ranges of the two waves will be $2D\left(\frac{p}{57{\cdot}3}\right)^3 \sin\delta$, the range of the superior transit wave being the greater when δ has the same name as the latitude of the place, and the smaller when they have different names.

In like manner if $s' = \sigma \sin$ sun's declination, the range of the superior transit solar wave may be supposed to exceed the range of the mean diurnal solar wave by $2s'$ when the sun's declination has the same name as the latitude of the place, and to be the lesser when their names are different.

When the moon passes the meridian at noon, the lunar high water happening, as we have seen, about one hour before the solar high water, the range of the tide following the moon's upper transit will be increased by $2as'$ beyond the mean diurnal range when the sun's declination is north; therefore when the moon passes the meridian at noon in north declination and the sun is also in north declination, the range of the tide following the moon's upper transit will exceed its mean diurnal range by

$$2\left\{D\left(\frac{p}{57\cdot3}\right)^3 \sin\delta + as'\right\},$$

and when the declinations are south, the signs of $\sin\delta$ and s' must be changed from positive to negative.

When the moon passes the meridian at 2 P.M. the sun's tide will have fallen about one hour when the lunar wave reaches its highest, and will therefore produce the same effect on the difference of the two ranges as when the moon passed the meridian at noon.

When the moon passes the meridian at 18^h, the lunar high water will happen one hour before, and when she passes the meridian at 20^h, one hour after, the low water of the inferior transit solar wave, which will therefore add $2as'$ to the difference of the ranges when the sun's declination has the same name as the latitude of the place. Therefore, if we denote by E the excess of the range of the superior transit wave over the mean diurnal range we shall have the following expressions for the value of E.

When the moon passes the meridian at 0^h, 2^h, 18^h or 20^h

$$E = 2\left\{D\left(\frac{p}{57\cdot3}\right)^3 \sin\delta + as'\right\} \quad \text{.....................(1)}$$

When she passes the meridian at 1^h or 19^h

$$E = 2\left\{D\left(\frac{p}{57\cdot3}\right)^3 \sin\delta + s'\right\} \quad \text{.....................(2)}$$

When the moon passes the meridian at 3^h, 17^h, 21^h, or 23^h

$$E = 2\left\{D\left(\frac{p}{57\cdot3}\right)^3 \sin\delta + bs'\right\} \quad \text{.....................(3)}$$

When the moon passes the meridian at 4^h, 10^h, 16^h, or 22^h

$$E=2\left\{D\left(\frac{p}{57\cdot3}\right)^3\sin\delta\right\} \qquad\qquad (4)$$

When the moon passes the meridian at 5^h, 9^h, 11^h, or 15^h

$$E=2\left\{D\left(\frac{p}{57\cdot3}\right)^3\sin\delta-bs'\right\} \qquad\qquad (5)$$

When the moon passes the meridian at 6^h, 8^h, 12^h, or 14^h

$$E=2\left\{D\left(\frac{p}{57\cdot3}\right)^3\sin\delta-as'\right\} \qquad\qquad (6)$$

When the moon passes the meridian at 7^h or 13^h

$$E=2\left\{D\left(\frac{p}{57\cdot3}\right)^3\sin\delta-s'\right\} \qquad\qquad (7)$$

sin δ being positive when the moon's declination is north and negative when south, also s' being positive when the sun's declination is north and negative when south.

The first line in Table IX. gives the probable value of E derived from observation when the moon passed the meridian at 22^h with horizontal parallax $57'\cdot6$ and declination 16° N. Introducing these values in equation (4) we have

$$8\cdot1 \text{ in.}=2D\left(\frac{57\cdot6}{57\cdot3}\right)\sin 16° \qquad\qquad (8)$$

In like manner from the second line Table IX. we have

$$8\cdot6 \text{ in.}=2\left\{D\left(\frac{57\cdot1}{57\cdot3}\right)^3\sin 19°+bs'\right\} \qquad\qquad (9)$$

and taking each line of the table in succession we obtain the following equations.

$$9\cdot25 \text{ in.}=2\left\{D\left(\frac{56\cdot4}{57\cdot3}\right)^3\sin 20°+as'\right\} \qquad\qquad (10)$$

$$10 \qquad =2\left\{D\left(\frac{55\cdot7}{57\cdot3}\right)^3\sin 20°+s'\right\} \qquad\qquad (11)$$

$$9\cdot9 \qquad =2\left\{D\left(\frac{55\cdot1}{57\cdot3}\right)^3\sin 18°+as'\right\} \qquad\qquad (12)$$

$$8\cdot75 \qquad =2\left\{D\left(\frac{54\cdot6}{57\cdot3}\right)^3\sin 15°+bs'\right\} \qquad\qquad (13)$$

$$6\cdot9 \qquad =2\left\{D\left(\frac{54\cdot3}{57\cdot3}\right)^3\sin 11°\right\} \qquad\qquad (14)$$

$$4{\cdot}1\text{ in.} = 2\left\{D\left(\frac{54{\cdot}2}{57{\cdot}3}\right)^3 \sin 5° - bs\right\} \quad \text{(15)}$$

$$0 = 2\left\{D\left(\frac{54{\cdot}5}{57{\cdot}3}\right)^3 \sin 1° + as'\right\} \quad \text{(16)}$$

$$3{\cdot}75 = 2\left\{D\left(\frac{55{\cdot}1}{57{\cdot}3}\right)^3 \sin 6°{\cdot}7 + s'\right\} \quad \text{(17)}$$

$$6{\cdot}6 = 2\left\{D\left(\frac{56{\cdot}0}{57{\cdot}3}\right)^3 \sin 12° + as'\right\} \quad \text{(18)}$$

$$9{\cdot}75 = 2\left\{D\left(\frac{57{\cdot}0}{57{\cdot}3}\right)^3 \sin 16° + bs'\right\} \quad \text{(19)}$$

$$8{\cdot}6 = 2\left\{D\left(\frac{58{\cdot}0}{57{\cdot}3}\right)^3 \sin 18°{\cdot}7\right\} \quad \text{(20)}$$

$$9{\cdot}4 = 2\left\{D\left(\frac{58{\cdot}7}{57{\cdot}3}\right)^3 \sin 20° + bs'\right\} \quad \text{(21)}$$

$$10{\cdot}1 = 2\left\{D\left(\frac{59{\cdot}4}{57{\cdot}3}\right)^3 \sin 20° + as'\right\} \quad \text{(22)}$$

$$10 = 2\left\{D\left(\frac{59{\cdot}9}{57{\cdot}3}\right)^3 \sin 18°{\cdot}4 + s'\right\} \quad \text{(23)}$$

$$8{\cdot}4 = 2\left\{D\left(\frac{60{\cdot}2}{57{\cdot}3}\right)^3 \sin 15°{\cdot}5 + as'\right\} \quad \text{(24)}$$

$$6{\cdot}5 = 2\left\{D\left(\frac{60{\cdot}1}{53{\cdot}7}\right)^3 \sin 11° + bs'\right\} \quad \text{(25)}$$

$$4{\cdot}1 = 2\left\{D\left(\frac{59{\cdot}9}{57{\cdot}3}\right)^3 \sin 5°{\cdot}7\right\} \quad \text{(26)}$$

$$1{\cdot}6 = 2\left\{D\left(\frac{59{\cdot}4}{57{\cdot}3}\right)^3 \sin 0°{\cdot}3 - bs'\right\} \quad \text{(27)}$$

$$2 = 2\left\{D\left(\frac{58{\cdot}7}{57{\cdot}3}\right)^3 \sin 5°{\cdot}4 + as'\right\} \quad \text{(28)}$$

$$5{\cdot}4 = 2\left\{D\left(\frac{58{\cdot}1}{57{\cdot}3}\right)^3 \sin 10°{\cdot}8 + s'\right\} \quad \text{(29)}$$

$$8{\cdot}1 = 2\left\{D\left(\frac{57{\cdot}5}{57{\cdot}3}\right)^3 \sin 15°{\cdot}4 + as'\right\} \quad \text{(30)}$$

$$10{\cdot}1 = 2\left\{D\left(\frac{56{\cdot}8}{57{\cdot}3}\right)^3 \sin 18°{\cdot}4 + bs'\right\} \quad \text{(31)}$$

$$11{\cdot} = 2\left\{D\left(\frac{56{\cdot}2}{57{\cdot}3}\right)^3 \sin 20°{\cdot}1\right\} \quad \text{(32)}$$

$$10{\cdot}4 = 2\left\{D\left(\frac{55{\cdot}6}{57{\cdot}3}\right)^3 \sin 20°{\cdot}1 + bs'\right\} \quad \text{(33)}$$

$$9{\cdot}5 \text{ in.} = 2\left\{D\left(\frac{55{\cdot}0}{57{\cdot}3}\right)^3 \sin 18^\circ{\cdot}5 + as'\right\} \ldots\ldots\ldots\ldots (34)$$

$$9{\cdot}1 = 2\left\{D\left(\frac{54{\cdot}5}{57{\cdot}3}\right)^3 \sin 15^\circ{\cdot}9 + s'\right\} \ldots\ldots\ldots\ldots (35)$$

$$7{\cdot}6 = 2\left\{D\left(\frac{54{\cdot}2}{57{\cdot}3}\right)^3 \sin 11^\circ{\cdot}8 + as'\right\} \ldots\ldots\ldots\ldots (36)$$

$$6{\cdot}75 = 2\left\{D\left(\frac{54{\cdot}1}{57{\cdot}3}\right)^3 \sin 6^\circ + bs'\right\} \ldots\ldots\ldots\ldots (37)$$

$$4{\cdot}9 = 2\left\{D\left(\frac{54{\cdot}2}{57{\cdot}3}\right)^3 \sin 0^\circ{\cdot}4\right\} \ldots\ldots\ldots\ldots (38)$$

$$-2{\cdot}1 = 2\left\{D\left(\frac{54{\cdot}6}{57{\cdot}3}\right)^3 \sin 5^\circ{\cdot}4 \times bs\right\} \ldots\ldots\ldots\ldots (39)$$

$$2{\cdot}1 = 2\left\{D\left(\frac{55{\cdot}3}{57{\cdot}3}\right)^3 \sin 10^\circ{\cdot}8 + as'\right\} \ldots\ldots\ldots\ldots (40)$$

$$5{\cdot}9 = 2\left\{D\left(\frac{56{\cdot}3}{57{\cdot}3}\right)^3 \sin 15^\circ{\cdot}2 + as'\right\} \ldots\ldots\ldots\ldots (41)$$

$$8{\cdot}5 = 2\left\{D\left(\frac{57{\cdot}4}{57{\cdot}3}\right)^3 \sin 18^\circ{\cdot}5 + as'\right\} \ldots\ldots\ldots\ldots (42)$$

$$8{\cdot}75 = 2\left\{D\left(\frac{58{\cdot}4}{57{\cdot}3}\right)^3 \sin 20^\circ + bs'\right\} \ldots\ldots\ldots\ldots (43)$$

To determine the most probable value of D from these equations, we first eliminate s', as', and bs'. To eliminate s'—subtracting

(41) from (11), $D\left\{\left(\frac{55{\cdot}7}{57{\cdot}3}\right)^3 \sin 20^\circ - \left(\frac{56{\cdot}3}{57{\cdot}3}\right)^3 \sin 15^\circ{\cdot}2\right\} = 2{\cdot}05$ in.

(29) (23), $D\left\{\left(\frac{59{\cdot}9}{57{\cdot}3}\right)^3 \sin 18{\cdot}4 - \left(\frac{58{\cdot}1}{57{\cdot}3}\right)^3 \sin 10{\cdot}8\right\} = 2{\cdot}3$

(17) (35), $D\left\{\left(\frac{54{\cdot}5}{57{\cdot}3}\right)^3 \sin 15{\cdot}9 - \left(\frac{55{\cdot}1}{57{\cdot}3}\right)^3 \sin 6{\cdot}7\right\} = 2{\cdot}675$

To calculate the coefficients of D we proceed as follows:

log 55·7 - - -	1·746	log 56·3 - - -	1·750
log 57·3 - - -	1·758	- - - - - -	1·758
difference - - -	$\bar{1}{\cdot}988$		$\bar{1}{\cdot}992$
	3		3
	$\bar{1}{\cdot}964$		$\bar{1}{\cdot}976$
log sin 20° - - -	9·534	log sin 15°·2 - -	9·419
log 0·315 - - -	$\bar{1}{\cdot}498$	log 0·249 - - -	$\bar{1}{\cdot}395$

$$\therefore \quad \left(\frac{55{\cdot}7}{57{\cdot}3}\right)^3 \sin 20^\circ \quad = 0{\cdot}315$$

$$\left(\frac{56{\cdot}3}{57{\cdot}3}\right)^3 \sin 15^\circ{\cdot}2 = 0{\cdot}249$$

$$\text{difference} \quad - \quad = 0{\cdot}066$$

Hence the first equation gives $0{\cdot}066 \times D = 2{\cdot}05$ in.
Second gives in like manner $0{\cdot}164 \times D = 2{\cdot}30$
Third gives - - - $0{\cdot}131 \times D = 2{\cdot}675$

To mean these according to their probable values multiply the second by 3 and the last by 2 and add the results to the first.

$$\begin{array}{ll} 1 & 0{\cdot}066 \times D = 2{\cdot}05 \text{ in.} \\ 3 & 0{\cdot}492 \times D = 6{\cdot}90 \\ 2 & 0{\cdot}262 \times D = 5{\cdot}35 \\ \hline 6 &)0{\cdot}820 \times D = 14{\cdot}3 \\ \hline \therefore & 0{\cdot}137 \times D = 2{\cdot}383 \quad \ldots\ldots\ldots\ldots\ldots(\sigma) \end{array}$$

gives the value of D derived from six of the equations after eliminating s'.

Equations (8), (14), (20), (26), (32), and (38) do not involve s', but the coefficient of D in (38) is so small as to render it practically of no value.

$$\begin{array}{ll} \text{Equation (8) gives} & 0{\cdot}279 \times D = 4{\cdot}05 \text{ in.} \\ (14) & 0{\cdot}163 \times D = 3{\cdot}45 \\ (20) & 0{\cdot}332 \times D = 4{\cdot}3 \\ (26) & 0{\cdot}113 \times D = 2{\cdot}25 \\ (32) & 0{\cdot}326 \times D = 5{\cdot}5 \end{array}$$

Multiply (8), (20), and (32) each by 3, (14) by 2, and (26) by 1.

$$\begin{array}{lll} \text{(8) multiplied by 3 gives} & & 0{\cdot}837 \times D = 12{\cdot}15 \text{ in.} \\ (20) & & 0{\cdot}996 \times D = 12{\cdot}90 \\ (32) & & 0{\cdot}978 \times D = 16{\cdot}5 \\ (14) & 2 & 0{\cdot}326 \times D = 6{\cdot}9 \\ (26) & 1 & 0{\cdot}113 \times D = 2{\cdot}25 \\ \hline & & 12)3{\cdot}250 \times D = 50{\cdot}7 \\ \hline & \therefore & {\cdot}271 \times D = 4{\cdot}225 \end{array}$$

Combining this with equation (σ) in proportion of the multipliers of D in each, or nearly so, multiply the above equation by 2, add equation (σ) and divide by 3, we have

$$\begin{array}{rl} & 0{\cdot}542 \times D = 8{\cdot}45 \text{ in.} \\ (\sigma) & 0{\cdot}137 \times D = 2{\cdot}383 \\ \hline 3) & 0{\cdot}679 \times D = 10{\cdot}833 \\ \hline & 0{\cdot}226 \times D = 3{\cdot}611 \ldots\ldots\ldots\ldots\ldots\ldots(\gamma) \end{array}$$

This is the most probable value of D derived from the twelve equations which do not involve either as' or bs'.

Next eliminate as' between the twelve equations involving it.

Subtract (30)	from (22)	gives	$0{\cdot}113 \times D = 1$ in.
(18)	(10)		$0{\cdot}129 \times D = 1{\cdot}7$
(36)	(34)		$0{\cdot}106 \times D = 0{\cdot}95$
(40)	(42)		$0{\cdot}15 \times D = 3{\cdot}2$
(28)	(12)		$0{\cdot}178 \times D = 3{\cdot}95$
(16)	(24)		$0{\cdot}296 \times D = 4{\cdot}2$

Multiplying each of the first three of these equations by 4, the fourth by 5, the fifth by 6, and the last by 10, we have

$$\begin{array}{rl}
 & 0{\cdot}452 \times D = \ \ 4 \text{ in.} \\
 & 0{\cdot}516 \times D = \ \ 6{\cdot}8 \\
 & 0{\cdot}424 \times D = \ \ 3{\cdot}8 \\
 & 0{\cdot}750 \times D = 16 \\
 & 1{\cdot}068 \times D = 23{\cdot}7 \\
 & \underline{2{\cdot}960 \times D = 42} \\
33) & \underline{6{\cdot}170 \times D = 99{\cdot}3} \\
 & 0{\cdot}187 \times D = \ \ 2{\cdot}918 \ldots\ldots\ldots\ldots(\alpha)
\end{array}$$

This gives the most probable value of D derived from twelve equations after eliminating as'.

In like manner eliminating bs' between the 12 equations involving it.

Subtracting (13)	from (33)	we have	$0{\cdot}090 \times D = 0{\cdot}825$ in.
(25)	(21)		$0{\cdot}148 \times D = 1{\cdot}45$
(37)	(43)		$0{\cdot}274 \times D = 1$
(39)	(9)		$0{\cdot}242 \times D = 5{\cdot}35$
(27)	(31)		$0{\cdot}321 \times D = 5{\cdot}85$
(15)	(19)		$0{\cdot}346 \times D = 5{\cdot}925$

Multiplying the first of these by 3, the second by 5, the third by 9, the fourth by 8, and the fifth and sixth by 11, adding and dividing by 47,

$$\begin{array}{rll}
3 & \text{give as follows,} & 0{\cdot}270 \times D = \ \ 2{\cdot}475 \text{ in.} \\
5 & & 0{\cdot}740 \times D = \ \ 7{\cdot}25 \\
9 & & 2{\cdot}466 \times D = \ \ 9 \\
8 & & 1{\cdot}936 \times D = \ 42{\cdot}8 \\
11 & & 3{\cdot}531 \times D = \ 64{\cdot}35 \\
\underline{11} & & \underline{3{\cdot}806 \times D = \ 65{\cdot}175} \\
47 & & \underline{)12{\cdot}749 \times D = 191{\cdot}05} \\
 & & 0{\cdot}265 \times D = \ \ 4{\cdot}065 \ldots\ldots\ldots(\beta)
\end{array}$$

This gives the most probable value of D which can be obtained from the twelve equations involving bs'. This equation must therefore be combined with equations (γ) and (a) in proportion of the coefficients of D in each.

$$\begin{array}{llrl} (\gamma) \text{ multiplied by 11 gives} & & 2{\cdot}486 \times D = & 39{\cdot}721 \text{ in.} \\ (a) & 9 & 1{\cdot}683 \times D = & 26{\cdot}262 \\ (\beta) & 13 & 3{\cdot}445 \times D = & 52{\cdot}845 \\ \text{Adding these,} & & 7{\cdot}614 \times D = & 118{\cdot}825 \end{array}$$

$$\therefore D = \frac{118{\cdot}825 \text{ in.}}{7{\cdot}614 \text{ in.}} = 15{\cdot}6 \text{ in.}$$

gives each of the thirty-six equations its fair effect in estimating the value of D.

To determine the value of s' from the thirty equations involving it, we must combine them so as to get the largest coefficient for s' and the smallest possible for D.

$$\begin{array}{lrlrr} \text{Equation} & (9) \text{ gives} & bs' = & 4{\cdot}3 \text{ in.} & -0{\cdot}324 \times D \\ & (21) & bs' = & 4{\cdot}7 & -0{\cdot}369 \times D \\ & (33) & bs' = & 5{\cdot}2 & -0{\cdot}314 \times D \\ & (43) & bs' = & 4{\cdot}375 & -0{\cdot}362 \times D \\ \text{Adding these,} & & 4bs' = & 18{\cdot}575 & -1{\cdot}369 \times D \end{array}$$

$$\begin{array}{lrlrr} \text{Equation} & (15) \text{ gives} & bs' = & -2{\cdot}050 \text{ in.} & +0{\cdot}074 \times D \\ & (13) & bs' = & 4{\cdot}375 & -0{\cdot}224 \times D \\ & (19) & bs' = & 3{\cdot}875 & -0{\cdot}272 \times D \\ & (25) & bs' = & 3{\cdot}250 & -0{\cdot}221 \times D \\ & (27) & bs' = & -0{\cdot}800 & +0{\cdot}014 \times D \\ & (31) & bs' = & 5{\cdot}050 & -0{\cdot}307 \times D \\ & (37) & bs' = & 3{\cdot}675 & -0{\cdot}088 \times D \\ & (39) & bs' = & -1{\cdot}050 & -0{\cdot}082 \times D \\ \text{Adding,} & & 8bs' = & 16{\cdot}325 & -1{\cdot}106 \times D \\ \text{but} & & 4bs' = & 18{\cdot}575 & -1{\cdot}369 \times D \end{array}$$

Subtracting the latter from the former,

$$4bs' = -2{\cdot}250 \quad +0{\cdot}263 \times D$$

Equation (10) gives		$as' =$	$4{\cdot}625$	$-0{\cdot}326 \times D$
	(22)	$as' =$	$5{\cdot}050$	$-0{\cdot}382 \times D$
	(24)	$as' =$	$4{\cdot}200$	$-0{\cdot}311 \times D$
	(42)	$as' =$	$4{\cdot}250$	$-0{\cdot}319 \times D$
Adding,		$4as' =$	$18{\cdot}125$	$-1{\cdot}338 \times D$
Equation (12) gives		$as' =$	$4{\cdot}950$	$-0{\cdot}275 \times D$
	(16)	$as' =$		$-0{\cdot}015 \times D$
	(18)	$as' =$	$3{\cdot}300$	$-0{\cdot}197 \times D$
	(28)	$as' =$	$1{\cdot}000$	$-0{\cdot}097 \times D$
	(30)	$as' =$	$4{\cdot}050$	$-0{\cdot}269 \times D$
	(34)	$as' =$	$4{\cdot}750$	$-0{\cdot}280 \times D$
	(36)	$as' =$	$3{\cdot}800$	$-0{\cdot}174 \times D$
	(40)	$as' =$	$1{\cdot}050$	$-0{\cdot}169 \times D$
Adding,		$8as' =$	$22{\cdot}900$	$-1{\cdot}476 \times D$
but		$4as' =$	$18{\cdot}125$	$-1{\cdot}338 \times D$
Taking the difference		$4as' =$	$4{\cdot}775$	$-0{\cdot}138 \times D$
From equation (11) we have		$s' =$	$5{\cdot}000$	$-0{\cdot}315 \times D$
	(23)	$s' =$	$5{\cdot}000$	$-0{\cdot}360 \times D$
Adding,		$2s' =$	$10{\cdot}000$	$-0{\cdot}675 \times D$
Equation (17) gives		$s' =$	$1{\cdot}875$	$-0{\cdot}104 \times D$
	(29)	$s' =$	$2{\cdot}700$	$-0{\cdot}196 \times D$
	(35)	$s' =$	$4{\cdot}550$	$-0{\cdot}235 \times D$
	(41)	$s' =$	$2{\cdot}950$	$-0{\cdot}249 \times D$
Adding,		$4s' =$	$12{\cdot}075$	$-0{\cdot}784 \times D$
but		$2s' =$	$10{\cdot}000$	$-0{\cdot}675 \times D$
Taking the difference		$2s' =$	$2{\cdot}075$	$-0{\cdot}109 \times D$
we have also		$4as' =$	$4{\cdot}775$	$-0{\cdot}138 \times D$
		$4bs' =$	$-2{\cdot}250$	$+0{\cdot}263 \times D$
Adding,		$2(1+2a+2b)\,s' =$	$4{\cdot}6$	$-0{\cdot}016 \times D$
$\therefore$		$(1+2a+2b)\,s' =$	$2{\cdot}3$	$-0{\cdot}008 \times D$

where $a = \frac{36}{41}$, $b = \frac{23}{41}$, and $D = 15{\cdot}6$ in.

$$\because \quad \frac{159}{41}+s'=2\cdot175 \text{ in.}$$

$$\therefore \quad s'=\frac{2\cdot175\times41}{159}=0\cdot56.$$

During the observations from which the above value of s' is derived the sun's declination averaged about 23° N., and

putting $$s'=\sigma \sin 23^\circ$$
$$\sigma=0\cdot56\times \operatorname{cosec} 23^\circ,$$
$$=2\cdot27 \text{ in.}$$

Hence the range of the lunar wave when the moon is on the equator, and the horizontal parallax 57′·3, is at the place of observation 4 feet 8 inches. This we have called the mean diurnal lunar range. An increase of one minute in the moon's horizontal parallax causes an increase of 3·124 inches in the range of the lunar wave. A change in the moon's declination also causes an alteration in the lunar range. For the sake of simplicity we have separated it into two, one depending on the cosine of the moon's declination, and the other on its sine. The first is expressed approximately by 60 in. × versin moon's declination, by which the ranges of both the waves are reduced; and the second by 31·2 in. × sin moon's declination, by which the upper transit lunar range is increased and the lower transit range diminished when the declination is north or the same name as the latitude of the place, and contrariwise when the moon's declination is south.

The solar wave though much smaller is similar to the lunar wave, and follows the solar transits by intervals about one hour greater than the mean luni-tidal interval, so that when the moon passes the meridian at 1^h or 13^h the solar and lunar high waters happen together.

The mean diurnal solar range, when the sun's declination was 23° N., we find to be 8·2 inches; and supposing the sun's declination to have a proportional effect on the solar range to that which the moon's declination has on the lunar range, when the sun is on the equator the solar range will be increased to 9 inches; consequently when the sun's declination is d° the solar range will be 9 in. $(1-\text{versin } d)$. Upon these data the following tables were calculated.

Table X.

PARALLAX CORRECTION TO BE APPLIED TO 4 FT. 8 IN. THE MEAN LUNAR RANGE.

Moon's Horizontal Parallax.	Correction.	Moon's Horizontal Parallax.	Correction.
53′·3	−12·5 in.	58′·3	+ 3·1 in.
54 ·3	9·4	59 ·3	6·2
55 ·3	6·2	60 ·3	9·4
56 ·3	3·1	61 ·3	12·5

Table XI.

DECLINATION CORRECTION TO BE APPLIED TO 4 FT. 8 IN. THE MEAN LUNAR RANGE.

Moon's Declination.	Moon's Upper Transit.			Moon's Lower Transit.		
	Horizontal Parallax.			Horizontal Parallax.		
	53′·3	57′·3	61′·3	53′·3	57′·3	61′·3
25° N.	+ 6·2 in.	+ 7·6 in.	+ 9·3 in.	−15·2 in.	−18·8 in.	−22·9 in.
20	5·7	7·1	8·7	11·6	14·3	17·4
15	4·9	6·0	7·3	8·2	10·1	12·3
10	3·6	4·5	5·5	5·1	6·3	7·7
5	2·0	2·5	3·0	2·3	2·9	3·5
0	0·0	0·0	0·0	0·0	0·0	0·0
5 S.	− 2·3	− 2·9	− 3·5	+ 2·0	+ 2·5	+ 3·0
10	5·1	6·3	7·7	3·6	4·5	5·5
15	8·2	10·1	12·3	4·9	6·0	7·3
20	11·6	14·3	17·4	5·7	7·1	8·7
25	15·2	18·8	22·9	6·2	7·6	9·3

Table XII.

CORRECTION FOR THE EFFECT OF THE MEAN SOLAR RANGE.

Moon's Meridian Passage.	Correction.	Moon's Meridian Passage.	Correction.	Moon's Meridian Passage.	Correction.
h.	in.	h.	in.	h.	in.
0	+7·9	4	0·0	8	−7·9
1	9·0	5	−5·0	9	5·0
2	7·9	6	7·9	10	0·0
3	5·0	7	9·0	11	+5·0

Table XIII.

CORRECTION TO RANGE DEPENDING ON THE SUN'S DECLINATION.

Moon's Meridian Passage.	Sun's Declination.					
	23°		20°		10°	
	North.	South.	North.	South.	North.	South.
h.	in.	in.	in.	in.	in.	in.
0	+0·2	−1·8	+0·3	−1·5	+0·2	−0·6
1	0·3	1·9	0·4	1·6	0·3	0·7
2	0·2	1·8	0·3	1·5	0·2	0·6
3	−0·2	1·4	0·0	1·2	0·1	0·5
4	0·0	0·0	0·0	0·0	0·0	0·0
5	−1·4	−0·2	−1·1	0·0	−0·5	+0·1
6	1·8	+0·2	1·5	+0·3	0·6	0·2
7	1·9	0·3	1·6	0·4	0·7	0·3
8	1·8	0·2	1·5	0·3	0·6	0·2
9	1·4	−0·2	1·1	0·0	0·5	0·1
10	0·0	0·0	0·0	0·0	0·0	0·0
11	−1·4	−0·2	−1·1	0·0	−0·5	+0·1
12	1·8	+0·2	1·5	+0·3	0·6	0·2
13	1·9	0·3	1·6	0·4	0·7	0·3
14	1·8	0·2	1·5	0·3	0·6	0·2
15	1·4	−0·2	1·1	0·0	0·5	0·1
16	0·0	0·0	0·0	0·0	0·0	0·0
17	−0·2	−1·4	0·0	−1·1	+0·1	−0·5
18	+0·2	1·8	+0·3	1·5	0·2	0·6
19	0·3	1·9	0·4	1·6	0·3	0·7
20	0·2	1·8	0·3	1·5	0·2	0·6
21	−0·2	1·4	0·0	1·1	0·1	0·5
22	0·0	0·0	0·0	0·0	0·0	0·0
23	−0·2	−1·4	0·0	−1·1	+0·1	−0·5

The following observations extracted from the tide register, compared with the times of high water and the ranges calculated from the foregoing tables, will serve to estimate their accuracy.

1st August 1864.—High water time, 7^h 13^m A.M.; height, 5 ft. 11 in.; range, 3 ft. 9 in. 31st July.—Moon's lower transit, 10^h 55^m; horizontal parallax, 54′·6; declination, 17°·4 N.

5th August.—High water time, 9^h 30^m A.M.; height, 6 ft. 2 in.; range, 4 ft. 0 in. 4th August.—Moon's lower transit, 13^h 51^m; horizontal parallax, 53′·95; declination, 3°·7 N.

7th August.—High water time, 10^h 37^m A.M.; height, 6 ft. 3 in.; range, 4 ft. 3 in. 6th August.—Moon's lower transit, 15^h 14^m; horizontal parallax, 54′·3; declination, 5°·3 S.

9th August.—High water time, $0^h\ 7^m$ P.M.; height, 6 ft. 0 in.; range, 3 ft. 9 in. 8th August.—Moon's lower transit, $16^h\ 43^m$; horizontal parallax, $55'\cdot25$; declination, $12°\cdot9$ S. At high water the wind was blowing fresh from the N.W., but changed rapidly round to the S.E. before the following low water and became light.

15th August.—High water time, $5^h\ 33^m$ A.M.; height, 6 ft. 0 in.; range, 4 ft. 1 in. 14th August.—Moon's upper transit, $9^h\ 38^m$; horizontal parallax, $60'\cdot15$; declination, $17°\cdot8$ S.

17th August.—High water time, $7^h\ 20^m$ A.M.; height, 6 ft. 9 in.; range, 5 ft. 8 in. 16th August.—Moon's upper transit, $11^h\ 34^m$; horizontal parallax, $61'\cdot3$; declination, $10°\cdot5$ S.

20th August.—High water time, $9^h\ 46^m$ A.M.; height, 7 ft. 5 in.; range, 6 ft. 6 in. 19th August.—Moon's upper transit, $14^h\ 22^m$; horizontal parallax, $60'\cdot5$; declination, $5°\cdot1$ N.

23rd August.—High water time, $0^h\ 3^m$ P.M.; height, 6 ft. 8 in.; range, 5 ft. 0 in. 22nd August.—Moon's upper transit, $17^h\ 3^m$; horizontal parallax, $57'\cdot0$; declination, $17°\cdot1$ N.

29th August.—High water time, $6^h\ 6^m$ A.M.; height, 5 ft. 11 in.; range, 3 ft. 4 in. 28th August.—Moon's lower transit, $9^h\ 40^m$; horizontal parallax, $54'\cdot3$; declination, 15° N.

31st August.—High water time, $7^h\ 23^m$ A.M.; height, 6 ft. 1 in.; range, 4 ft. 1 in. 30th August.—Moon's lower transit, $11^h\ 7^m$; horizontal parallax, $53'\cdot95$; declination, $8°\cdot3$ N.

At time of moon's

lower transit,	31st July 1864,	sun's	declination	18°	N.
„	4th August	„	„	16 ·9	
„	6th „	„	„	16 ·4	
„	8th „	„	„	15 ·8	
upper transit,	14th „	„	„	14 ·1	
„	16th „	„	„	13 ·8	
„	19th „	„	„	12 ·4	
„	22nd „	„	„	11 ·3	
lower transit,	28th „	„	„	9 ·4	
„	30th „	„	„	8 ·6	

The necessary lunar and solar elements are given for the convenience of the reader.

To find from the tables the time of high water and the range of the A.M. tide on the 1st August 1864,

We have

Mean luni-tidal interval, - - - - - -	$7^h\ 36^m{\cdot}8$
Correction from Table III. for moon's lower transit, $10^h\ 55^m$, -	$+18\ {\cdot}1$
" IV. for moon's horizontal parallax, 54′·6, -	$+12\ {\cdot}4$
" VI. for moon's declination, 17°·4 N., - -	$+\ 6\ {\cdot}6$
" VII. - -	$0\ {\cdot}0$
Luni-tidal interval, - - - - - - -	$8^h\ 14^m$
31st July—Moon's lower transit, - - -	10 55
High water, 1st August, - - - - - -	$7^h\ \ 9^m$ A.M.
" " by observation, - -	7 13

Mean lunar diurnal range of tide, - - -	4 ft. 8 in.
Correction from Table IX., moon's horiz. parallax, 54′·6, - - - -	− 8·4
" X., moon's declin., 17°·4 N., lower transit, - -	−10·6
" XI., solar moon's lower transit, $10^h\ 55^m$, -	+ 4·7
" XII., sun's declination, 18° N.,	− 0·7
Calculated range, - - - - - - -	3 ft. 5 in.
Observed range, - - - - - - -	3 9

For the 5th August A.M. high water we have

Mean luni-tidal interval, - - - - - -	$7^h\ 36^m{\cdot}8$
Correction from Table III., for moon's low. transit, $13^h\ 51^m$, - - - -	$-\ 9\ {\cdot}4$
" IV., for moon's horizontal parallax, 53′·95, -	$+10\ {\cdot}7$
" VI., for moon's declination, 3°·7 N., - - - -	$+\ 1\ {\cdot}3$
" VII., {moon's low. transit, $1^h\ 51^m$ A.M., sun's declination, 16°·9, N., }	$+\ 1\ {\cdot}6$
Luni-tidal interval, - - - - - -	$7^h\ 41^m$
4th August—Moon's lower transit, - - -	13 51
High water, 5th August, - - - - -	$9^h\ 32^m$ A.M.
" " observed, - - -	9 30

Mean lunar diurnal range, - - - - -	4 ft. 8 in.
Correction from Table IX., moon's horizontal parallax, 53′·95, - - -	−11
" X., moon's declin., 3°·7 N., lower transit, - -	− 1·8
" XI., moon's meridian passage, 1^h 51^m, - - -	− 8·1
" XII., sun's declination, 16°·9 N., - - - -	− 1·2
Range of tide, 5th August A.M., - - -	4 ft. 2 in.
" " by observation, -	4 0

For the 7th August we have

Mean luni-tidal interval, - - - - - -	7^h 36^m·8
Correction from Table III., moon's lower transit, 15^h 14^m, - - -	−19 ·8
" IV., moon's hor. parallax, 54′·3, - - -	+ 8
" IV., moon's declination, 5°·3 S., - - -	− 1 ·5
" VII., {moon's low. transit, 3^h 14^m A.M., sun's declination, 16°·4 N.,}	+ 3 ·4
Luni-tidal interval, - - - - - - -	7^h 27^m
6th August—Moon's lower transit, - - -	15 14
High water, 7th August, - - - - -	10^h 41^m A.M.
" " by observation, - -	10 37

Mean diurnal lunar range, - - - - -	4 ft. 8 in.
Correction from Table IX., moon's hor. parallax, 54′·3, - - -	−9·4
" X., moon's declin. 5°·3 S., lower transit, - -	+2·2
" XI., Moon's merid. passage, 3^h 14^m, - - -	+3·7
" XII., sun's declination, 16°·4 N., - - -	−0·7
Range of A.M. tide, 7th August, - - -	4 ft. 4 in.
" observed range, - - -	4 3

9th August—Mean luni-tidal interval, - -	$7^h\ 36^m{\cdot}8$
Correction from Table III., Moon's lower transit, $16^h\ 43^m$, - - - -	-23
„ IV., moon's hor. parallax, $55'{\cdot}25$, - - - -	$-\ 5$
„ VI., moon's declination, $12^\circ{\cdot}9$ S., - - - -	$-\ 4\ {\cdot}2$
„ VII., {moon's low. transit, $4^h\ 43^m$ A.M., sun's declination, $15^\circ{\cdot}8$ N., }	$+\ 3\ {\cdot}8$
Luni-tidal interval, - - - - - - -	$7^h\ 18^m$
8th August—Moon's lower transit, - - -	16 43
9th August—High water, - - - - -	$0^h\ 1^m$ P.M.
„ „ observed time, - -	0 7

Mean diurnal lunar range, - - - - -	4 ft. 8 in.
Correction from Table IX., moon's hor. parallax, $56'{\cdot}25$, - - -	$-6{\cdot}2$
„ X., moon's declin. $12^\circ{\cdot}9$ S., lower transit, - -	$+4{\cdot}9$
„ XI., moon's merid. passage, $16^h\ 43^m$, - - -	$-3{\cdot}7$
„ XII., sun's declination, $15^\circ{\cdot}8$,	0
Calculated range, 9th August A.M., - - -	4 ft. 3 in.
Observed „ „ - - -	3 9

15th August—Mean luni-tidal interval, - -	$7^h\ 36^m{\cdot}8$
Correction from Table III., moon's upper transit, $9^h\ 38^m$, - - -	$+26\ {\cdot}2$
„ IV., moon's hor. parallax, $65'{\cdot}15$, - - -	$-14\ {\cdot}2$
„ VI., moon's declination, $17^\circ{\cdot}8$ S., - - -	$+\ 9\ {\cdot}4$
„ VII., sun's declination, -	0
	$7^h\ 58^m$
14th August—Moon's upper transit, - - -	9 38
High water, 15th August, - - - - -	$5^h\ 36^m$ A.M.
Observed time, - - - - - - -	5 33

Mean lunar diurnal range, - - - - -	4 ft. 8 in.
Correction from Table IX., moon's hor. parallax, 60′·15, - - -	+ 9
„ X., moon's declin., 17°·8 S., upper transit, - -	−14·4
„ XI., moon's merid. passage, 9^h 38^m, - - -	− 2
„ XII., sun's declination, 14°·1 N., - - -	0
Calculated range, 15th August, - - - -	4 ft. 1 in.
Observed „ „ - - - - -	4 1

17th August—Mean luni-tidal interval, - -	7^h 36^m·8
Correction from Table III., moon's merid. passage, 11^h 34^m, - - -	+12 ·2
„ IV., moon's hor. parallax, 61′·3, - - -	−17 ·2
„ VI., moon's declin., 10°·5, sun's upper transit, -	+5 ·7
„ VII., sun's declination, 13°·8,	0
Luni-tidal interval, - - - - - - -	7^h 37^m·5
Moon's upper transit, 16th August, - - -	11 34
High water, 17th August, - - - - - -	7^h 11^m·5 A.M.
„ „ by observation, - -	7 20

Mean diurnal lunar range, - - - - -	4 ft. 8 in.
Correction from Table IX., moon's horizontal parallax, 61′·3, - - -	+12·5
„ X., moon's declin., 10°·5, sun's upper transit, -	−8
„ XI., moon's merid. passage, 11^h 34^m, - - -	+6·5
„ XII., sun's declination, 13°·8 N., - - -	−0·8
Calculated range for 17th August, - - - -	5 ft. 6 in.
Observed „ „ - - - -	5 8

20th August—Mean luni-tidal interval, - -	$7^h\ 36^m{\cdot}8$
Correction from Table III., moon's upper transit, $14^h\ 22^m$, - - -	$-13\ {\cdot}5$
„ IV., moon's horizontal parallax, $60'{\cdot}5$, - -	$-9\ {\cdot}6$
„ VI., moon's declination, $5^\circ{\cdot}1$ N., - - -	$-2\ {\cdot}2$
„ VII., sun's declination, $12^\circ{\cdot}4$ N, - - -	$+1\ {\cdot}7$
Luni-tidal interval, - - - - - -	$7^h\ 13^m$
19th August—Moon's upper transit, - - -	14 22
High water, 20th August, - - - -	$9^h\ 35^m$ A.M.
„ „ by observation, - -	9 46

Mean diurnal lunar range, - - - - -	4 ft. 8 in.
Correction from Table IX., moon's horizontal parallax, $60'{\cdot}5$, - -	+10
„ X., moon's declin., $5^\circ{\cdot}1$, upper transit, - -	+ 3
„ XI., moon's meridian passage, $14^h\ 22^m$, - -	+ 6·9
„ XII., sun's declination, $12^\circ{\cdot}4$ N., - - - - -	− 0·8
Calculated range, A.M., 20th August, - - -	6 ft. 3 in.
Observed „ „ - - -	6 6

23rd August—Mean luni-tidal interval, - -	$7^h\ 36^m{\cdot}8$
Correction from Table III., moon's upper transit, $17^h\ 3^m$, - - - -	$-22\ {\cdot}8$
„ IV., moon's horizontal parallax, $57'{\cdot}8$, - -	$-1\ {\cdot}2$
„ VI., moon's declin., $17^\circ{\cdot}1$, N. upper transit, -	$-6\ {\cdot}8$
„ VII., sun's declin., $11^\circ{\cdot}3$ N.,	$+2\ {\cdot}7$
Luni-tidal interval, - - - - - - -	$7^h\ 9^m$
Moon's upper transit, 22nd August, - - -	17 3
High water, 23rd August, - - - - -	$0^h\ 12^m$ P.M.
„ „ by observation, - -	0 3

Mean diurnal lunar range, - - - - -	4 ft. 8 in.
Correction from Table IX., moon's horizontal parallax, 57′·8, - -	+1·6
„ X., moon's declin., 17°·1, upper transit, - -	+6·7
„ XI., moon's meridian passage, 17^h 3^m, - -	−5·0
„ XII., sun's declination, 11°·3 N., - - -	0·0
Calculated range for 23rd August, - - -	4 ft. 11 in.
Observed „ „ - - -	5 0

29th August—Mean luni-tidal interval, - -	7^h 36^m·8
Correction from Table III., moon's lower transit, 9^h 40^m, - - -	+25 ·9
„ IV., moon's horizontal parallax, 54°·3, - -	+15
„ VI., moon's declination, 15° N., - - -	+ 6 ·2
„ VII., sun's declination, 9°·4 N., - - -	0
Luni-tidal interval, - - - - - - -	8^h 24^m
28th August, moon's lower transit, - - -	9 40
High water, 29th August, - - - -	6^h 4^m A.M.
„ „ by observation, - -	6 6

Mean lunar diurnal range, - - - - -	4 ft. 8 in.
Correction from Table IX., moon's horizontal parallax, 54′·3, - -	−9·4
„ X., moon's declin., 15° N., lower transit, - -	−8·7
„ XI., moon's meridian passage, 9^h 40^m, - -	−1·6
„ XII., sun's declination, 9°·4 N., - - -	0·0
Calculated range for 29th August, A.M., - -	3 ft. 0 in.
Observed „ „ - -	3 4

31st August—Mean luni-tidal interval, - -	$7^h\ 36^m{\cdot}8$
Correction from Table III., moon's lower transit, $11^h\ 7^m$, - - -	+16 ·5
" IV., moon's horizontal parallax, 53′·95, - -	+14 ·6
" VI., moon's declin., 8°·3 N., lower transit, - -	+ 4 ·2
Luni-tidal interval, - - - - - - -	$8^h\ 12^m$
Moon's lower transit, 30th August, - - -	11 7
High water, 31st August, - - - - - -	$7^h\ 19^m$ A.M.
" " by observation, - -	7 23
Mean diurnal lunar range, - - - - -	4 ft. 8 in.
Correction from Table IX., moon's horizontal parallax, 53′·95, - -	−10·5
" X., moon's declin., 8°·3 N., lower transit, - -	− 4·5
" XI., moon's meridian passage, $11^h\ 7^m$, - -	+ 5·3
Calculated range for 31st August, - - -	3 ft. 10 in.
Observed " " - - -	4 1

To find the time of the low water following on high water—To $6^h\ 13^m$ apply the declination correction given in Table VI. with a contrary sign, and add the result to the time of the preceding high water already calculated; this will give the time of low water sufficiently near for practical purposes. To 3 ft. 9½ in., the average height of the mean tide line, add half the range for the height of high water, and subtract half the range for the height of the low water.

Half the sum of the times of high and its following low water will give the time of mean tide when the height is 3 ft. 9½ in.

If the height of the average mean tide line above the sounding zero is noted on the chart, the depth of water to be expected at any time at any place on the chart can be ascertained sufficiently near for all practical purposes from six weeks' observations.

I will now give a general description of the tide gauges constructed on board H.M.S. *Columbia* in 1844 and 1845, mentioned in page 272.

A cylindrical copper float of large circular area, with conical ends, the axes being all in the same straight line, loaded so as to float in water with its axis vertical when one of its conical ends was downwards and part of the cylinder immersed, was placed in a vertical tube in which it could move freely up and down, and into which the sea water was admitted through an aperture in its bottom of about one inch area. To the vertex of the upper conical end a chain was attached which passed over a pulley movable on a horizontal axis; to the other end of the chain, after passing over the pulley, a weight was hung, which tended to pull the chain and pulley round and raised the float in the water, so that only about one half of the cylinder and the lower conical end remained immersed. Two lengths of chain of the same size and weight as that attached to the float were hung below the counter-balance weight, and, with it were placed in another vertical tube with a closed end at the bottom, so that the weight moved freely up and down the tube with part of chain below it always resting on the bottom. The links of the chain were all made exactly the same size, and so that twenty-four links of the chain, when it was stretched straight, measured exactly one foot, and each link added exactly half an inch to the length of the chain. On the circumference of the pulley twelve hollows were made exactly equal to each other and equidistant, separated by flat spaces, on and into which the chain exactly fitted, so that when the chain was put over the pulley with each alternate link in a hollow, with the intermediate links lying flat on the circumference between them, and the pulley was turned alternately in opposite directions, the same links always came in the same hollows or on the same flat spaces, as the case might be. When the pulley made exactly one revolution twenty-four links or twelve inches of chain passed off the pulley, and the float either rose or fell one foot, as the case might be. The horizontal axis of the pulley rested on two bearings, separated by a distance depending on the range of the tide; between the bearings the axis of the pulley was enlarged to a cylinder of one inch diameter, on the surface of which the thread of a screw was grooved out, the horizontal distance between two consecutive threads of which was exactly one inch. A small metal tube, about one inch long, capable of sliding easily along the cylindrical axis of the pulley, had a small pin fixed on its inside at the middle point; the pin was made to fit easily into the hollow of the thread of the screw; when the tube was placed over the axis of the pulley with its pin in the groove, as the pulley moved the metal tube slid along its axis, in such a manner that one revolution of the pulley made it move hori-

zontally exactly one inch ; the metal tube carried a pencil or marker passing through a small vertical tube on its side, and this pressed upon the upper circumference of a large cylinder which revolved uniformly once in twelve hours.

We may remark that the sea water being admitted through a small aperture in the bottom of the tube the surface of the water in it rose and fell with the tide as fast as it did outside; whilst the wind waves and other fluctuations of short period of the surface outside the tube did not affect the motion of the surface inside.

The two lengths of chain hung below the counter-balance weight caused the tension of the chain at the upper end of the float to remain the same in all positions of the float.

CHAPTER XI.

TO REDUCE A PLANE TO A MERCATORIAL PROJECTION.

WHEN surveying operations extend over large areas, a difficulty arises from the fact that the surface of the earth cannot be laid out flat on a plane.

The earth is found to have very nearly the shape of an oblate spheroid generated by the revolution of an ellipse of small eccentricity about its minor axis, which coincides with the axis of the earth.

A conical surface whose axis coincides with the earth's will touch the surface of the earth in a parallel of latitude very nearly equal to the angle of the cone. A short distance north or south of the line of contact the two surfaces will so nearly coincide with each other, that normals to the earth's surface, not far from the touching line, produced to meet the surface of the cone, will meet it in points whose bearings and distances from each other will differ insensibly from those of the points of the earth's surface from which they are derived.

A plan of the earth's surface delineated on a narrow strip of the surface of the tangent cone according to the above law can be cut along lines parallel to, and at short equal distances north and south of the line of contact, and the part so cut off can be laid out flat on a plane. When this is done, the line of contact, which is the middle latitude of the sheet, will be part of the circumference of a circle whose radius is equal to the distance of the vertex of the cone from the line of contact of the two surfaces, the centre of the circle being towards the elevated pole, and its radii meridians.

In the above construction the meridians and their tangents at the middle latitude of the sheet are supposed to be so close to each other through the whole breadth of the sheet north and south, that no sensible error will be introduced by considering them coincident. When the difference of latitude between the north and south edges of the sheet is so large that on the scale of projection there is a sensible difference between the length of the meridian and its tangent at the middle

latitude of the sheet, comprised within the limits of the sheet, this plan is not applicable.

The working sheets of a nautical survey generally lie well within this limit; but each sheet generally having a different middle latitude from the others, will lie in a plane differing slightly from the planes of the others. In order to bring the whole survey into one plane the different sheets must be reduced to a Mercatorial projection, unless the latitude is within the polar circle.

To reduce a plane sheet to a Mercatorial projection proceed as follows.

Let *OPQR* (Fig. 59) be a plane sheet, *A*, *B*, and *C* the three main or principal points on it. The bearings and distances of *A*, *B*, and *C* from each other ought to be accurately determined and noted on the sheet, as well as their latitudes and longitudes.

The true bearing of *B* from *A* is the angle which the meridian through *A* makes with the straight line *AB*. Draw the straight line *AB* with great care, and produce it both ways to the edge of the paper; suppose $m \times 5$ is the multiple of 5 inches which is nearest to the number of inches *AB* is long; take $5m$ inches off the brass scale with a pair of beam compasses, place one leg of the compasses on *A*, and see if the point of the other leg when placed on *AB* towards *B* lies well inside the edge of the sheet, that is to say, not less than $1\frac{1}{2}$ inches from the edge of the paper, and that the circular arc traced by it round *A* towards the left hand as far as the meridian through *A* does so too; if not, the radius must be reduced to $(m-1)\ 5$ inches. Having thus determined the best radius to be used for projecting the meridian through *A* from the straight line *AB*, find the chord of the true bearing angle corresponding to it, and in the usual way find the point through which and *A* the meridian passes. In like manner, by means of the true bearing of *C* from *A*, and with the same radius as before, determine the same point. The two projections must agree. Although the first is to be preferred on account of its chord being the shorter, the second serves as a check. If the two projections do not agree, and the discrepancy arises from an inaccuracy in the angle *BAC*, the whole sheet must be reprojected. But this is not likely to happen if the main triangle *ABC* was carefully projected at first in the manner recommended in Chapter IX. Having satisfactorily determined the point through which and *A* the meridian must pass, draw through them by means of a steel straight edge the meridian N_1AS_1 right across the sheet.

Having thus drawn the meridian through *A*, in a similar

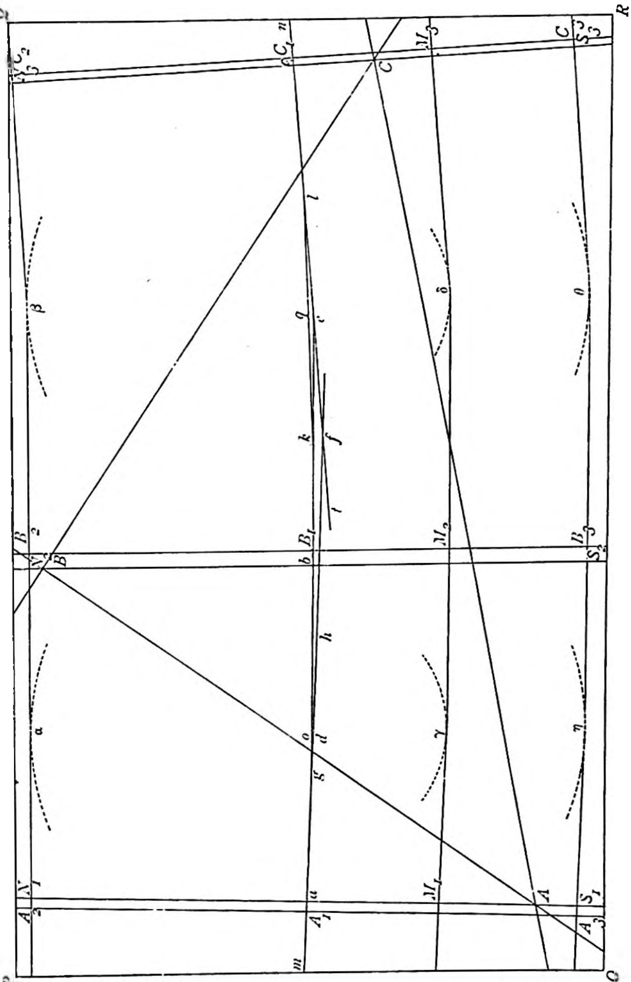


FIG. 59.

manner draw through B the meridian N_2BS_2, and through C the meridian N_3CS_3. Too much care cannot be bestowed on drawing these meridians, as all the others, as well as the parallels of latitude, will depend upon them.

With the brass scale and a pair of beam compasses measure on the inch scale the lengths of AB, BC, and CA. Add their lengths thus found together, and let S_i be the sum. Add together the lengths of AB, BC, and CA expressed in miles, and let S_m be the sum. From the logarithm of S_i subtract the logarithm of S_m; the difference will be the logarithm of the number of brass scale inches to a mile of distance. Call this log q.

Let L be the nearest latitude to the middle latitude of the sheet that can be expressed by an exact mile. Take the differences between L and the latitudes of A, B, and C respectively; to the logarithms of each of these differences add the logarithm of q, and let a, b, and c be the natural numbers corresponding to the respective sums of the logarithms. With the beam compasses take off an inch from the brass scale, place one leg of the compasses over A and the other on the line AN_1, so that the point of the beam compass leg may be just clear of the paper, press the point of the compass into A, the other point will mark the meridian AN at the point a in which the circle of latitude L cuts it. In the same manner by taking b and c inches respectively from the brass scale with the beam compasses, the points b and c in which the parallel of latitude L cuts the meridians N_2BS_2 and N_3CS_3 respectively are determined. Through b draw the straight line dbe perpendicular to the meridian N_2BS_2. Through a draw the straight line mad perpendicular to the meridian N_1AS_1, cutting dbe in the point d; and through e draw the straight line $ncef$ perpendicular to the meridian N_3CS_3, cutting dbe in e, and mad produced in f. Then, if the projections have been well and accurately made, da should be equal to db, be to ec, and af to fe. This is a very severe test, which is seldom exactly satisfied. When the inequalities are large, there are two sources of error from which they may arise, which must be examined with care.

First, the meridians N_1AS_1, N_2BS_2, and N_3CS_3 may not have been drawn with sufficient accuracy, or the perpendiculars to them, that is, ad, dbe, and cef may be in error. Produce the straight line cef to t, then the angle dft should be equal to the inclination of the meridians N_1AS_1, N_3CS_3 to each other. If this condition is satisfied, try if the angle bdf is equal to the inclination of the meridian N_1AS_1 to N_2BS_2. If this is also the case, the difference must arise from a want of agreement in

the latitude given by the points a, b, and c; the lengths Aa, Bb, and Cc should be recalculated and remeasured with every care, and if no error can be discovered the positions of a, b, and c must each be corrected as follows, on the supposition that the points A, B, and C have been equally well projected, and also their meridians; and we therefore assume the errors in Aa, Bb, and Cc to be proportional to their lengths—viz., that ax is the error of Aa, bx that of Bb, and cx that of Cc. Let i be the inclination of the meridians N_1AS_1 and N_3CS_3 to each other expressed in circular measure, and $d=\frac{af-fc}{2}$, then since A and C are the same side of the middle latitude circle we shall have very approximately

$$(a-c)\,x=di \text{..........................}(1)$$

then if

$$\frac{ad-db}{2}=d_1,$$

and the inclination of the meridians N_1AS_1 and N_2BS_2 be i_1 remembering that A and B are on opposite sides of the middle latitude line, we have

$$(a+b)\,x=d_1i_1 \text{..........................}(2)$$

similarly

$$(a+c)\,x=d_2i_2 \text{..........................}(3)$$

where $d_2=\frac{be-ce}{2}$ and $i_2=i-i_1$. Each of these equations will give a value of x, and in order to obtain its most probable value from the three multiply each by its coefficient of a, and add hence

$$\{(a-c)^2+(a+b)^2+(b+c)^2\}x=(a-c)i+(a+b)i_1+(b+c)i_2$$

and

$$x=\frac{(a-c)di+(a+b)d_1i_1+(b+c)d_2i_2}{(a-c)^2+(a+b)^2(b+c)^2} \text{..................}(4)$$

where $i_1+i_2=i$. Since i seldom exceeds 0·004, unless d, d_1, and d_2 are all large, and all have the same sign, this correction will be practically of no importance, and will seldom need to be applied, unless the sheets and the scale are very large.

The positions of the points d and e being satisfactorily settled, referring to Figure 60, where the inclinations of the meridians are enormously exaggerated in order to show the *modus operandi* more clearly, mad, dbe, and ecn are the sides of a polygon touching the circle of middle latitude L in the points a, b, and c respectively. Bisect ad in g, db in h, be in k,

and ec in l; join gh and kl by straight lines which bisect respectively in the points o and q. The circle of latitude L will touch gh in o, and kl in q, and the polygon $aghklc$ will coincide more nearly with the circle of latitude L than the polygon $mden$ does; and by continuing the process we can draw a polygon differing insensibly from the circle of latitude L. When this is done, which a very few bisections will accomplish, draw the sides of the polygon (which practically is the circumference of the circle of middle latitude) carefully in ink.

Let D be the difference of longitude expressed in decimals of a mile between A and the next full mile of longitude to the

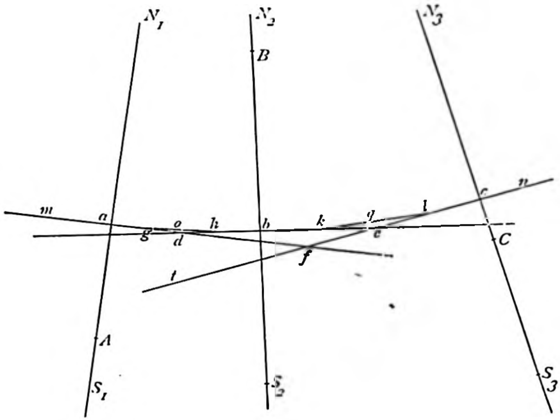


FIG. 00.

west of A; to $\log D$ add $\log \cos L$ and $\log q$. The natural number corresponding to the sum of these logarithms will be the number of brass scale inches from the point a, at which the meridian first next to A, corresponding to an exact mile of longitude, cuts the circle of latitude L. Take this length from the brass scale with a pair of compasses and lay it off on the circle of middle latitude from a towards the west; let A_1 be the point thus determined, and through it draw the straight line $A_3A_1A_2$ parallel to N_1AS_1 right across the sheet from north to south, or *vice versa;* $A_3A_1A_2$ will represent the mile meridian next to the west of A without sensible error. In like manner draw the mile meridian $B_3B_1B_2$ which is nearest to B, and $C_3C_1C_2$ the mile meridian next to the east of C.

Let δ be the difference in decimals of a mile between the latitude of A and the next full mile of latitude to the southward of A; to log δ add log q. The natural number corresponding to their sum will be the number of brass scale inches, measured from A towards the south on the meridian N_1AS_1, at which the mile parallel of latitude next to the south of A cuts it. Take this length from the brass scale with a pair of compasses and lay it off from A on the meridian AS_1 towards the south, and let S_1 be the point thus determined. With a pair of beam compasses take off very carefully the length of aS_1, make bS_2 and cS_3 each equal to aS_1, and from the centres o, q, etc., describe with radius equal to aS_1 circular arcs at η and θ, etc.; draw $S_1\eta$ and $S_2\eta$ touching the circular arc at η, and the straight lines $S_2\theta$ and $S_3\theta$ touching the circular arc θ at θ, etc. The line $A_3S_1\eta S_2\theta S_3$... will represent with sufficient accuracy the mile parallel of latitude next to the south of A. In a similar manner, by means of B's latitude, draw $N_1\alpha N_2\beta N_3$... to represent the mile parallel of latitude next to the north of B, and by means of C's latitude $\gamma M_2\delta M_3$ to represent the mile parallel of latitude nearest to C.

Subdivide each of the portions of these four parallels of latitude between the meridians $A_2A_1A_3$ and $B_2B_1B_3$ into as many equal parts as there are miles of longitude between them, and also the portions between $B_2B_1B_3$ and $C_2C_1C_3$ in a similar manner. The length of a mile of longitude on the same parallel given by the space between each two meridians ought to be equal, and by comparing them we have another check on the accuracy of the work. Being satisfied on this point, draw each mile meridian carefully through the four points corresponding to it, which should all be in the same straight line. Measure a mile of longitude to the west of $A_2A_1A_3$ if the sheet extends so far on each of the parallels, and draw the mile meridian next west of $A_2A_1A_3$, etc. Similarly draw the mile meridian next to the east of $C_2C_1C_3$, etc., and so on. To draw the parallels of latitude one mile distant from each other and the four parallels already drawn between which they lie, subdivide S_1M_1 into as many equal parts as there are miles of latitude between the two parallels of latitude passing through the points S_1 and M_1 respectively; and in a similar manner M_1a into as many equal parts as there are miles of latitude between the parallels passing through M_1 and a; and also subdivide N_1a into as many equal parts as there are miles of latitude between a and N_1. In like manner subdivide the meridians N_2BbS_2 and N_3cCS_3; next take the meridians which pass nearest to the points o and q, etc., in which the polygon of middle latitude $aobqc$ touches the parallel of middle latitude, and subdivide them in the same way

and join the points on the meridians thus determined by straight lines parallel to the sides of the four parallels of latitude already drawn, thus filling the sheets with polygonal parallels of latitude one mile of latitude distant from each other. These, intersecting the meridians drawn one mile of longitude from each other, will cover the sheet with trapeziums differing so slightly from oblongs that on the scale of projection the difference is too small to be measurable; of these the longer sides are each equal to a mile of latitude, and the shorter sides to a mile of longitude.

A Mercatorial sheet corresponding in latitude and longitude with the plane sheet must also be prepared as follows. A

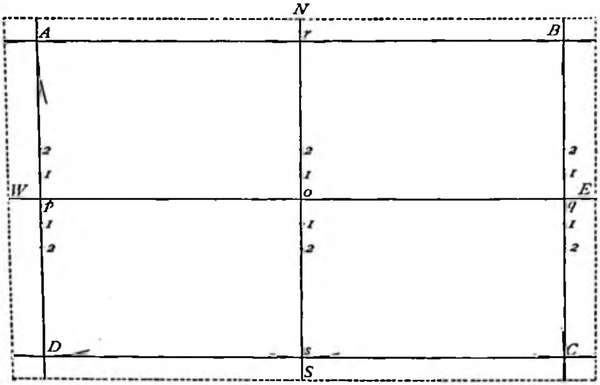


FIG. 61.

sheet of sufficient size must be laid flat on the table, the steel straight edge laid flat upon it across the centre of the sheet parallel to its edges. Weights having been distributed evenly along the steel, it should be pushed slowly towards one of the edges, to which it must be kept parallel, and when the sheet commences to pucker between the scale and the nearer edge of the paper, which generally happens with uncut paper about an inch or so from its edge, with a sharp knife cut the puckered part off by running the knife along the outer edge of the steel. The other edges of the sheet must then be treated in the same way. Every sheet should be treated in this way before commencing to project on it, and after that no important point should be placed or line used for projection within one inch or more from the edge of

the paper. The sheet being thus prepared, through O its middle point draw the straight lines WOE, NOS (Fig. 61), perpendicular to each other, and as nearly parallel respectively to the edges of the sheet as possible, and let WOE represent the parallel of latitude, L the middle latitude of the plane sheet, or in other words to be the projection on it of the plane sheet parallel of latitude $mA_1aobB_1qcC_1n$, and take NOS for the projection of the mile meridian which is nearest to the middle of the plane sheet.

Let M be the middle latitude of the survey and s the number of inches to a mile in this latitude which gives the scale of the Mercatorial projection of the same. Let m be the invariable length of a mile of Mercatorial longitude, then

$$m = s \cos M\left(1 + \frac{\cos^2 M}{149}\right)$$
$$= s \cos M \sec^2\psi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in a form adapted to logarithmic computation by putting $\tan^2\psi = \frac{\cos^2 M}{149}$.

Find the approximate length in inches of WOE, and suppose it to be n inches, and that $m \times 2^p < n$, whilst $m \times 2^{p+1} > n$. Lay the brass scale on the paper just below WOE, and with the beam compasses take off very carefully from it $m \times 2^p$ inches, and make Op and Oq respectively each equal to the length just taken off the scale, and by successive bisections divide pO and Oq into lengths each equal to m inches.

To 0·301030 add the logarithm of m and the log secant of L, the middle latitude of the plane sheet, and call the sum log S, and from it subtract successively the log secants of $(L+1')$, $(L-1')$, $(L+3')$, $(L-3')$, $(L+5')$, $(L-5')$, etc. Take out the natural numbers corresponding to the differences of these logarithms in succession, and let n_1, n_2, n_3, etc., correspond to $(L+1')$, $(L+3')$, $(L+5')$, etc., and s_1, s_2, s_3, etc., correspond to $(L-1')$, $(L-3')$, $(L-5')$, etc. Suppose $n_1+n_2+\ldots+n_r = N$ inches be less than ON, whilst $N+n_{r+1}$ is greater than ON. With a pair of beam compasses take N inches carefully from the brass scale, and set another pair of beam compasses accurately to the length of Op or Oq. On the meridian ON set off $Or = N$ inches, and then placing one leg of the same beam compasses at p describe a circular arc at A, and from q as a centre with the same radius describe a circular arc at B. With the other pair of beam compasses from the centre r, with the distance equal to Op or Oq, describe two fine circular arcs, cutting the former just described in the points A and B re-

spectively. In like manner supposing $s_1+s_2+\ldots+s_r=S$ to be less than OS, while $S+s_{r+1}$ is greater than OS, lay off the point s on the meridian OS so that $Os=S$, and then the points D and C in which the parallel of latitude through s cuts the meridians drawn through the points p and q respectively.

Lay a steel straight edge, with its edge passing through the points A and B. If the table and steel are good, as well as the construction, the edge will also pass through the point r, in which case draw the straight line ArB right across the sheet. In like manner draw the straight lines BqC, CsD, and DpA through each of the three points defining them, taking care always to lay the steel ruler inside the line so that the line may be between the ruler and the edge of the paper.

With a pair of compasses take from the brass scale n inches, and on the meridians pA, ON, and qB lay off p_1, O_1, and q_1, each equal to n_1, then make p_2, O_2, and q_2, each equal to n_1+n_2, and so on until we arrive at the points p_{r-1}, O_{r-1}, and q_{r-1}. The points 1 will be those in which the parallel of latitude $L+2'$ cuts the meridians pA, Or, and qB respectively; points 2 those in which the parallel of latitude $L+4'$ cuts them, and so on. Through the points 1, 2, and $r-1$, draw straight lines right across the paper, and thus the part of the sheet to the north of WOE will be covered with parallels of latitude each two miles of Mercatorial latitude from the two between which it is drawn. If the parts of the three meridians pA, ON, and qB, which lie between any two of these parallels, be bisected, we shall obtain the parallel of latitude situated between them with sufficient accuracy, and then drawing the straight lines through these points of bisection the sheet to the north of WOE will be filled with parallels of latitude corresponding to each mile from latitude L to latitude $L+2r'$. In a similar manner the parallels of latitude one mile distant from each other are drawn to the southward of WOE.

Through the points already determined on WOE, where each mile meridian cuts it respectively, draw straight lines parallel to the meridians DpA, NOS, or BqC, already drawn, which will cover the sheet with oblongs whose shorter sides are equal to a mile of longitude, whilst their longer sides are miles of latitude respectively to each of these oblongs. A trapezium on the plane sheet differing, as we have seen, insensibly from an oblong corresponds in latitude and longitude and has its sides proportional thereto; the part of the plane sheet within these four lines is reduced by the aid of proportional compasses into the oblong on the Mercatorial sheet, and thus the plane projection is transformed to a Mercatorial projection.

CHAPTER XII.

ON THE VARIATIONS OF A SHIP'S COMPASS.

THIS subject is of great importance, because it frequently happens that the ship's compass is the only means by which the directions we wish to know can be estimated, and therefore the more accurately the variations of the compass have been determined the nearer to the truth the true bearings of objects and the ship's course, estimated from her compass course and bearings, will be.

The variation of a ship's compass changes as she moves from one place to another, and at the same place with every alteration in the position of her head: the latter change is called the deviation of the compass. I have found in practice the more simple mode is to consider the variation of the ship's compass as fluctuating and to tabulate it for the different positions of the ship's head at the place of observation. When the ship changes her place, the consequent change in the variations of the compass is applied as a constant correction to each variation taken from the table.

The smaller the deviation of the ship's compass the better, especially in vessels employed in surveying, and it is therefore very necessary to seek for the spots in a vessel where the deviation is a minimum, and, if practicable, to place the standard compass there. In wooden vessels this can always be done.

There are several ways in which the variations of a ship's compass can be determined.

First, when a small rock or island can be found, round which the vessel can steam so as to be visible in all directions from a spot in it; at this place set up a theodolite and determine the true bearing of its zero point. The vessel should then steam slowly round the island in sight of the theodolite, and near enough to enable it to be seen from her standard compass, and keeping it as nearly as possible on her starboard beam. At the instant given by a preconcerted signal from the ship the observer at the theodolite cuts off the ship's standard compass,

noting the *time* and the reading of the theodolite; at the same instant the observer at the standard compass takes the bearing of the theodolite, noting it, the direction of the ship's head, and the time. The ship's head should be steady when the signal is made, and one observation should be taken with the ship's head on every second point of the compass, or as near thereto as possible. When the circuit of the island is completed the vessel should bring the theodolite on her port beam, and, commencing when the ship's head is in the *opposite* direction to that it was when the first observation was made, steam slowly round the island in the opposite direction, taking similar observations when the ship changes the direction of her head two points or nearly so, in the same way as in the first round.

Thus, sixteen observations will be made, one with the ship's head on every second point of the compass whilst she is steaming round the island under port helm, and sixteen similar observations with the ship's head on the corresponding points of the compass whilst she is steering round under starboard helm.

The mean of the two observations when the ship is turning under opposite helms and her head at the same point of the compass, or nearly so, is taken, and from the variations so determined a curve of variation constructed.

TABLE I.

Time of Signal, 5th Sept., 1864.	Ship's Head by Standard Compass.	True Bearing of		Observed Bearing of Station from Ship.	Variation of Compass.	Remarks.
		Ship from Station observed.	Station from Ship deduced.			
h. m. s.		° ′	° ′	° ′	° ′	
8 32 10	South.	S. 69 10 W.	N. 69 10 E.	N. 87 40 E.	18 30 W.	Ship steaming slowly round the Theodolite Station about one mile distant, turning under starboard helm, with the Station nearly on her port beam.
36 15	S.S.E.	47 3	47 3	62 · 0	14 57	
40 30	S.E.	23 51	23 51	34 40	10 49	
44 30	E.S.E. $\frac{1}{8}$ E.	2 14	2 14	9 0	6 46	
48 50	E. $\frac{3}{8}$ N.	20 46 E.	20 46 W.	16 0 W.	4 46	
53 15	N.E. by E. $\frac{3}{4}$ E.	43 51	43 51	40 0	3 51	
57 5	N.E. by N. $\frac{1}{4}$ N.	64 5	64 5	52 0	12 5	
9 1 40	N. by E. $\frac{1}{8}$ E.	89 52	89 52	73 20	16 52	
5 35	N. $\frac{1}{4}$ W.	N. 69 59	S. 69 59	88 0	22 1	
10 0	N.N.W.	45 17	45 15	S. 73 20	28 3	
13 45	N.W.	24 16	24 16	58 0	33 44	
18 10	N.W. by W. $\frac{3}{4}$ W.	0 13	0 13	36 0	35 47	
22 20	W. $\frac{1}{2}$ N.	22 11 W.	22 11 E.	13 0	35 11	
27 0	W.S.W. $\frac{1}{16}$ W.	46 15	46 15	13 30 E.	32 45	
30 55	S.W. $\frac{1}{16}$ S.	67 7	67 7	40 0	27 7	
35 10	S. by W. $\frac{7}{8}$ W.	89 3	89 3	66 0	23 3	

TABLE II.

Time of Signal, 5th Sept., 1864.	Ship's Head by Standard Compass.	True Bearing of		Observed Bearing of Station from Ship.	Variation of Compass.	Remarks.
		Ship from Station observed.	Station from Ship deduced.			
h. m. s.		° ′	° ′	° ′	° ′	
9 50 50	N. $\frac{1}{8}$ E.	S. 68 28 W.	N. 68 28 E.	East.	22 31 W.	Ship steaming slowly round Station, about one mile distant, under port helm, keeping it nearly on her starboard beam.
56 35	N.N.E.	N. 80 41	S. 80 41	S. 66 0 E.	14 41	
10 1 0	N.E. $\frac{1}{16}$ E.	55 3	55 3	47 0	8 3	
6 30	E.N.E.	25 16	25 16	21 30	3 46	
10 15	E. $\frac{1}{8}$ S.	5 34	5 34	0 40	4 54	
14 10	S.E. by E. $\frac{7}{8}$ E.	16 28 E.	16 38 W.	23 0 W.	6 22	
17 45	S.E. $\frac{1}{4}$ S.	36 57	36 57	46 20	9 23	
21 10	S.S.E. $\frac{1}{8}$ E.	53 29	53 29	68 0	14 31	
24 30	S. $\frac{1}{8}$ W.	69 53	69 53	89 0	19 7	
28 10	S.S.W. $\frac{1}{4}$ W.	S. 89 5	N. 89 5	N. 65 0	24 5	
31 30	S.W.	75 11	75 11	46 20	28 51	
35 40	W.S.W.	54 14	54 14	22 30 W.	31 44	
39 35	W. $\frac{1}{4}$ S.	34 21	34 21	North.	34 21	
43 15	W.N.W. $\frac{1}{4}$ W.	16 44	16 44	N. 19 20 E.	34 6	
47 35	N.W. $\frac{1}{8}$ W.	8 58 W.	8 58 E.	42 30	33 32	
52 30	N.N.W.	37 23	37 23	65 40	28 17	

TABLE III.

Ship's Head.	Variation of Standard Compass.	Ship's Head.	Variation of Standard Compass.
	° ′		° ′
S. $\frac{1}{16}$ W.	18 48 W.	N. $\frac{1}{32}$ W.	22 16 W.
S.S.E. $\frac{1}{16}$ E.	14 44	N.N.W.	28 10
S.E. $\frac{1}{8}$ S.	10 6	N.W. $\frac{1}{32}$ N.	33 37
E S.E.	6 34	W.N.W.	36 0
E. $\frac{1}{4}$ N.	4 50	W. $\frac{1}{8}$ N.	34 46
N.E. by E. $\frac{7}{8}$ E.	3 48	W.S.W. $\frac{1}{32}$ W.	32 14
N. $3\frac{1}{16}$ E.	10 4	S.W. $\frac{1}{32}$ W.	27 59
N. $1\frac{1}{16}$ E.	15 46	S.S.W. $\frac{1}{16}$ W.	23 34

The foregoing examples will show how this is done. Table I. gives the observations made with the vessel turning under starboard helm; Table II., the observations made whilst steaming round under port helm; Table III., the mean of each pair of observations made with the ship turning under *opposite* helms, but with her head in the same direction or nearly so.

The first line in Table I. gives ship's head south, variation of

compass 18° 30′ W., the ship turning under starboard helm; the ninth line in Table II. gives ship's head S. $\frac{1}{8}$ W., variation 19° 7′ W., the ship turning under port helm. Taking the arithmetic mean of these two, we have ship's head S. $\frac{1}{16}$ W., variation of compass 18° 48′ W.; and proceeding in a similar manner we obtain the quantities inserted in Table III.

We proceed now to construct the curve of variation in the following manner. Through the middle of a sheet of blue lined paper draw the straight line *AB*, Figure 62, perpendicular to the blue straight lines; take it to denote variation of the compass 20° W., and take the blue straight lines to denote the position of the ship's head by the compass. The first blue straight line to the left hand, *CAD*, representing that the direction of the ship's head is south by compass; the next blue straight line to the right of *CAD* that the ship's head is S. by W., and so on; take $\frac{3}{16}$ inch to represent 1° variation of compass, the direction *AC* as positive westerly variation of the compass, and that of *AD* as negative westerly variation. The first line of Table III. gives ship's head S. $\frac{1}{16}$ W. variation of the standard compass 18° 48′ W. Subtracting this from 20° W., we have 1° 12′ or 72′; using the $\frac{3}{8}$ inch scale subdivided to twelfths, each subdivision will correspond to 10′ variation of compass. We therefore take 7·2 subdivisions from this scale with a pair of dividers, and with a pair of proportional compasses $\frac{1}{16}$ the distance between two consecutive blue lines, and on *AB* set off *Ap* equal to this distance, the point *p* will correspond to ship's head by compass S. $\frac{1}{16}$ W. draw *pq* parallel to *AD*, and make *pq* equal to the distance on the dividers representing 1° 12′ variation of compass; and, being in the negative direction, the point *q* will represent variation of the compass 18° 48′ W., and ship's head S. $\frac{1}{16}$ W. The last line of Table III. gives ship's head S.S.W. $\frac{1}{16}$ W., variation of compass 23° 34′ W. proceeding in a similar manner. We project the point *r* corresponding to this observation, and in the same way the points *s*, *t*, *v* ... *l*, *m*, projected in like manner, will represent the other observations recorded in the table. Lastly, the point *n*, bearing the same relation to *BF* that *q* bears to *AD*, is laid off to represent the first observation in Table III. repeated, join *qr*, *rs*, *st* ... *lm*, and *mn*, with fine straight pencil lines, and draw a fine ink curved line of gradual curvature to pass as nearly as possible through the points *q*, *r*, *s* ... *m*, *n* with the straight pencil lines for its chords, or as nearly so as possible, and so that the curvature may change gradually. This curve will represent the most probable values of the variations of the standard compass to be derived from the observations recorded in Table III. as the ship's head turns round the compass from

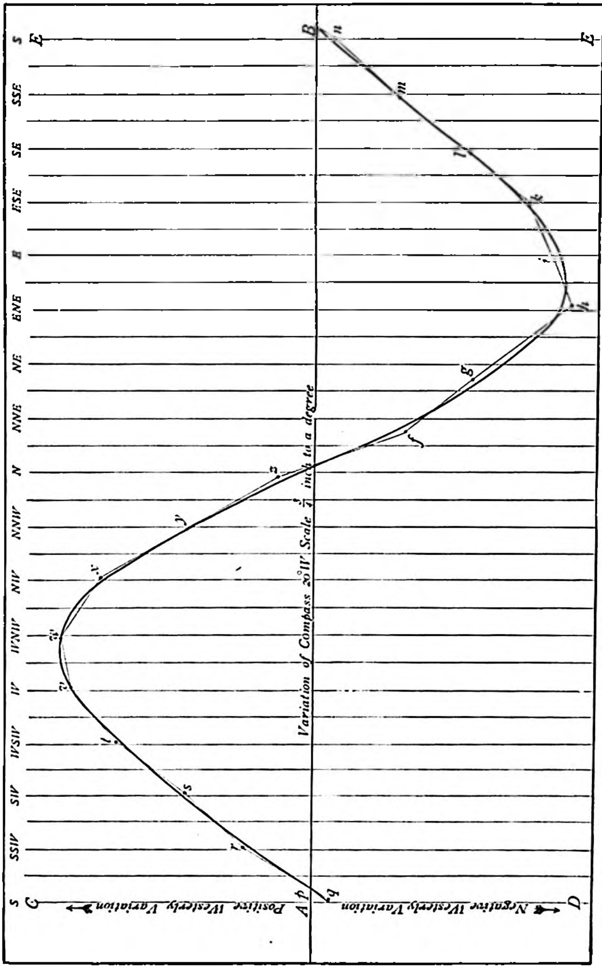

Variation of Compass 20° W Scale 1/4 inch to a degree
Positive Westerly Variation
Negative Westerly Variation
S
SSW
SW
WSW
W
WNW
NW
NNW
N
NNE
NE
ENE
E
ESE
SE
SSE
S

FIG. 62.

south until it reaches south again. Table IV. gives the variation of the compass for the ship's head on each point of the compass derived from this curve by measuring the length of its ordinate on each blue straight line and applying the value thus obtained with its proper sign to variation 20° W.

TABLE IV.

Ship's Head.	Variation of Compass.		Ship's Head.	Variation of Compass.	
	°	′		°	′
South	18	38 W.	North.	21	22 W.
S. by W.	21	20	N. by E.	17	15
S.S.W.	23	40	N.N.E.	13	34
S.W. by S.	25	58	N.E. by N.	10	50
S.W.	27	56	N.E.	8	24
S.W. by W.	30	0	N.E. by E.	6	6
W.S.W.	31	49	E.N.E.	4	20
W. by S.	33	30	E. by N.	4	4
West.	35	2	East.	4	50
W. by N.	35	48	E. by S.	5	52
W.N.W.	36	0	E.S.E.	6	45
N.W. by W.	35	3	S.E. by E.	8	32
N.W.	33	34	S.E.	10	38
N.W. by N.	31	5	S.E. by S.	12	40
N.N.W.	28	20	S.S.E.	14	36
N. by W.	24	55	S. by E.	16	46

Secondly, when a rock or island such as above cannot be found, the theodolite should be set up at a convenient place on the high line of the shore, where it can be approached by the ship sufficiently near for the observer at the standard compass to see the theodolite station distinctly, and the ship can have good clear water to turn in between bearings from the shore station, comprising an angle of not less than 60°; she then steers round to seaward of the theodolite station in oval curves between the limits of the extreme bearings of safety, turning first under starboard and then under port helm. In this manner a series of observations similar to the foregoing will be obtained, which, treated in the same manner, will give the most probable values of the variations of the compass.

Thirdly, when high distant peaks of land can be seen over the high line, a conspicuous object situated as near as possible to the high line of the coast, where the vessel can approach the shore with safety, should be selected, and the true bearing of two distant peaks from the selected object determined either by observations made at the selected object or by observa-

tions made on board the vessel when she is on the same straight line as that joining the selected object with the distant peak; the second distant peak should make an angle at the ship of about 30° with the other, and is selected in case the ship's rigging should intercept the view of the first peak from the observer at the standard compass when the ship's head is on a course for which an observation must be taken in this case. The other peak, being in sight, should have its magnetic bearing observed when it comes in line with the selected object. The vessel is then run across this line, known by the distant peak coming in one with the selected object, with her head on each point of the compass for which an observation is to be made—first, turning under starboard helm to bring her on her course; and, secondly, under port helm, taking care that she is turning *very* slowly at the instant she crosses the bearing line. When the peak is so distant that its true bearing from the ship does not sensibly alter as she steams round the same point with her head on the different points of the compass selected for observation, the near object used to define the line of true bearing may be dispensed with.

For example, in 1867 a beacon on a high distant peak in the island of Sicily bore from a station made on the high line of its coast N. 21° 3′ W. H.M.S. *Hydra* was run across this line of true bearing in the manner recommended. Table V. gives the mean of the two observations made when the ship's head was on the same course, or nearly so, but turning very slowly under opposite helms.

TABLE V.

Ship's Head.	Bearing of Beacon by Compass.	Variation of Compass.	Ship's Head.	Bearing of Beacon by Compass.	Variation of Compass.
	° ′	° ′		° ′	° ′
North.	10 30 W.	10 33 W.	S. $\frac{1}{8}$ W.	9 0 W.	12 3 W.
N.N.E.	10 50	10 13	S.S.W. $\frac{1}{4}$ W.	9 15	11 48
N.E.	12 15	8 48	S.W. $\frac{1}{4}$ W.	9 20	11 43
E.N.E.	11 20	9 43	W.S.W.	8 30	12 33
East.	10 40	10 23	West.	7 30	13 33
E.S.E.	11 15	9 48	W. by N. $\frac{3}{4}$ N.	8 0	13 3
S.E. $\frac{1}{8}$ E.	9 50	11 13	N.W.	7 50	13 23
S.S.E.	9 0	12 3	N.N.W.	8 0	13 3

To project the above, through the middle of a sheet of blue lined paper draw the straight line AB (Fig. 63) perpendicular to the blue straight lines, and take it to represent variation of

the standard compass 11° 30′ W., take the left hand blue straight line *NAC* to denote that the ship's head is north by the standard compass, the next blue straight line to the right of *NAC* that the ship's head is N. by E., and so on, the space between two consecutive blue lines representing one point of the compass, so that when we arrive at the right-hand blue straight line, *NBD*, the ship's head will have turned round the compass and arrived again at the north point. The direction *AN* is taken to represent positive westerly variation, and the opposite direction *AC* negative westerly variation, one inch representing a degree of variation.

Referring to Table V, we see that when the ship's head was north the observed variation was 10° 33′ W., subtracting this from 11° 30′ W., we have 57′ to be laid off from *A* in the negative westerly direction, *AC*, using the inch scale subdivided to twelfths, and remembering that eleven of these subdivisions represent 55′, take off with a pair of dividers the length corresponding to 57′, and from the point *A* lay this distance off on *AC*, and thus define the point *a*; then *Aa* will represent − 57′, and consequently the point *a* will be the projection of the first line of Table V., and expresses that when the ship's head was north the variation of the standard compass was 10° 33′ W.; and proceeding in a similar manner we project the points, *b*, *c*, *d*, ... *n*, *o*, *p*, representing the observations in Table V. as they follow the first. On *BD* take $Ba_2 = Aa$, then a_2 will be the first observation in Table V. repeated for the convenience of drawing the curve; join *ab*, *bc*, *cd*, ... *no*, *op*, pa_2, with fine straight pencil lines, and through the points in which these straight lines cut the intermediate blue lines, that is the blue line which is between the two points which the pencil line joins, draw another series of straight pencil lines joining each consecutive pair of the points so determined, and also through or as near as possible to these points draw with a free hand a curve of gradual curvature of which the direction on crossing each blue line shall be intermediate between that of the two straight pencil lines drawn from the two points between which it is situated, and approaching as near as possible to the points *a*, *b*, *c*, ... In this manner the ink curve in the Figure was drawn, the distances of the points in which this curve cuts each of the blue lines successively from the straight line *AC* were then measured; the values thus obtained applied to variation 11° 30′ W. gave the variations tabulated in the following.

TABLE VI.

Ship's Head.	Variation of Compass.	Ship's Head.	Variation of Compass.	Ship's Head.	Variation of Compass.
	° ′		° ′		° ′
North.	10 53 W.	S.E. by E.	10 34 W.	W.S.W.	12 33 W.
N. by E.	10 23	S.E.	11 15	W. by S.	13 4
N.N.E	9 57	S.E. by S.	11 40	West.	13 15
N.E. by N.	9 31	S.S.E.	11 56	W. by N.	13 15
N.E.	9 15	S. by E.	12 3	W.N.W.	13 15
N.E. by E.	9 15	South.	12 2	N.W. by W.	13 15
E.N.E.	9 43	S. by W.	11 56	N.W.	13 15
E. by N.	10 4	S.S.W.	11 50	N.W. by N.	13 11
East.	10 9	S.W. by S.	11 46	N.N.W.	12 49
E. by S.	10 6	S.W.	11 50	N. by W.	11 45
E.S.E.	10 10	S.W. by W.	12 5	North.	10 53

Examining Fig. 63, we see that the observations denoted by the points a, b, c, e, f, m, and n differ more than the others from the curve giving the most probable values of the variations of the compass that can be derived from the observations recorded in Table V., that c differs more than any of the others, and the difference amounts to 27′; this however is not greater than the error that might arise in taking an observation with the *Hydra's* standard compass. under such circumstances. Comparing Tables VI. and V. we find that on the average Table VI. gives 2′ more westerly variation than Table V., and that the mean of the differences taken irrespective of sign is 10′; we may therefore feel satisfied that Table VI. gave the variations of the *Hydra's* standard compass sufficiently near for all practical purposes.

The sun when its altitude is small affords a very good opportunity for determining the variations of a ship's compass. For this purpose she should be steered slowly round under a small helm, so that the points of the compass pass the lubber's line very slowly in succession. The bearing of the sun's centre should be observed with the standard compass the instant her head passes each point of the compass, and the time of each observation noted, with the bearing of the sun's centre and the direction of the ship's head. The altitude of the sun is observed, and the time of the watch noted at the beginning and end of the operation; and if the true bearing of the sun in the interval does not vary uniformly with the time or nearly so, the sun's altitude with the time must also be observed at such intermediate times between the first and last observa-

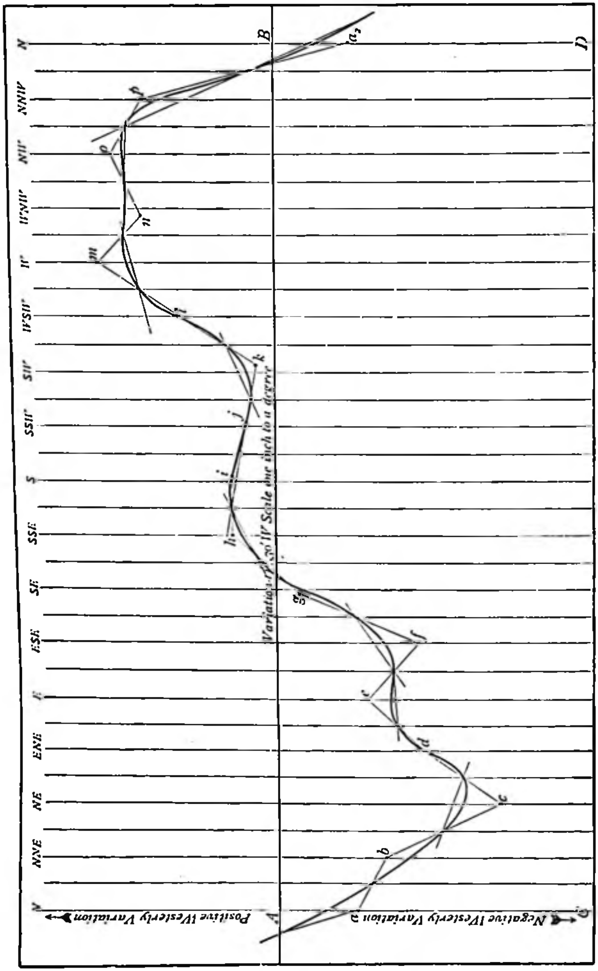

N
NNW
NW
WNW
W
WSW
SW
SSW
S
SSE
SE
ESE
E
ENE
NE
NNE
N
Positive Westerly Variation
Negative Westerly Variation
A
B
C
D
a
b
c
d
e
f
g
h
i
j
k
l
m
n
o
p

FIG. 63.

tions that the sun's true bearing between two consecutive observations of the sun's altitude may be taken to vary with the time without introducing any practical error. From the true bearings of the sun, calculated from the observed altitudes, its true bearing at the time of each observation of its magnetic bearing can be calculated, and the difference between the two bearings will give the variation of the compass when the ship's head is in the direction it had when the observation was made. The following example will explain the process more clearly.

H.M.S. *Hydra*, 2nd July, 1868, in latitude 26° 8′ S. watch fast of Greenwich mean time, 8^m 19^s, height of eye above the sea 18 feet, and error of sextant $-30''$. The sun's declination was 23° 0′ N.

Time by Watch.	Alt. Sun's LL.	True Bearing Sun's Centre.
h. m. s.	° ′ ″	° ′
4 22 32	6 50 20	N 60 11·5 W.
4 41 25	3 14 0	N 62 24 W.
4 50 20	1 31 0	N 63 27 W.

The interval between the two first observations is 18^m 53^s, and the change in the sun's true bearing, 2° 12′·5; or in one minute of time the average change in the sun's true bearing was 7′·02. The interval between the second and third observations is 8^m 55^s, and the change in the sun's true bearing during this 1° 2′; therefore, during the second interval, the change in the sun's true bearing averaged 6′·92 per minute of time. Consequently, we may take the sun's true bearing to change uniformly between the first and last observation of the sun's altitude at the rate of 7′ during a minute of time, and the true bearings calculated for any time between the observations on this supposition will be sufficiently near the truth for our purposes. In the following Table VII., the observations of the magnetic bearings are noted in degrees and tenths of a degree, as also are the calculated true bearings of the sun's centre, as well as the variations of the compass derived from them.

TABLE VII.

Time by Watch.			Ship's Head by Standard.	Bearing of Sun's Centre by Standard.	True Bearing of Sun's Centre.	Variation of Compass.
h	m	s		°	°	°
4	23	9	N.N.W.	N. 31·0 W.	N. 60·3 W.	29·3 W.
	24	43	N.W. by N.	30·8	60·4	29·6
	25	52	N.W.	30·0	60·6	30·6
	26	47	N.W. by W.	29·8	60·7	30·9
	27	45	W.N.W.	30·0	60·8	30·8
	28	46	W. by N.	31·0	60·9	29·9
	29	34	West.	31·0	61·0	30·0
	30	28	W. by S.	31·3	61·1	29·8
	31	30	W.S.W.	31·7	61·2	29·5
	32	29	S.W. by W.	32·5	61·4	28·9
	33	15	S.W.	33·0	61·5	28·5
	34	10	S.W. by S.	33·0	61·6	28·6
	35	3	S.S.W.	33·3	61·7	28·4
	35	48	S. by W.	33·0	61·8	28·8
	36	31	South.	32·3	61·9	29·6
	38	0	S. by E.	33·0	62·0	29·0
	39	4	S.S.E.	32·3	62·2	29·9
	41	25	S.E. by S.	33·0	62·4	29·4
	42	24	S.E.	33·8	62·5	28·7
	43	22	S.E. by E.	33·3	62·6	29·3
	44	15	E.S.E.	33·8	62·7	28·9
	45	5	E. by S.	34·0	62·8	28·8
	45	45	East.	34·3	62·9	28·6
	46	24	E. by N.	33·7	63·0	29·3
	46	53	E.N.E.	33·4	63·0	29·6
	47	15	N.E. by E.	33·3	63·1	29·8
	47	55	N.E.	32·7	63·2	30·5
	48	20	N.E. by N.	33·0	63·2	30·2
	49	0	N.N.E.	33·0	63·3	30·3
	49	54	N. by E.	33·3	63·4	30·1
	50	20	North.	33·0	63·4	30·4
	50	54	N. by W.	32·0	63·5	31·5

Fig. 64 shows the projections of these observations, made in the same way as those in Fig. 63, before described, 1 denoting the observation recorded in the first line of Table VII., 2, that in the second line, and so on to 32, which represents that in the last line of the Table. Join the points 1 and 2 with a straight pencil line, also the points 2 and 3, 3 and 4, and so on. If the observations are correct these straight lines will be the chords of the curve, showing the variations of the ship's compass; but the errors in the observations cause these straight lines to take an irregular zigzag course instead of the gradual change in their direction which the gradual change in the dis-

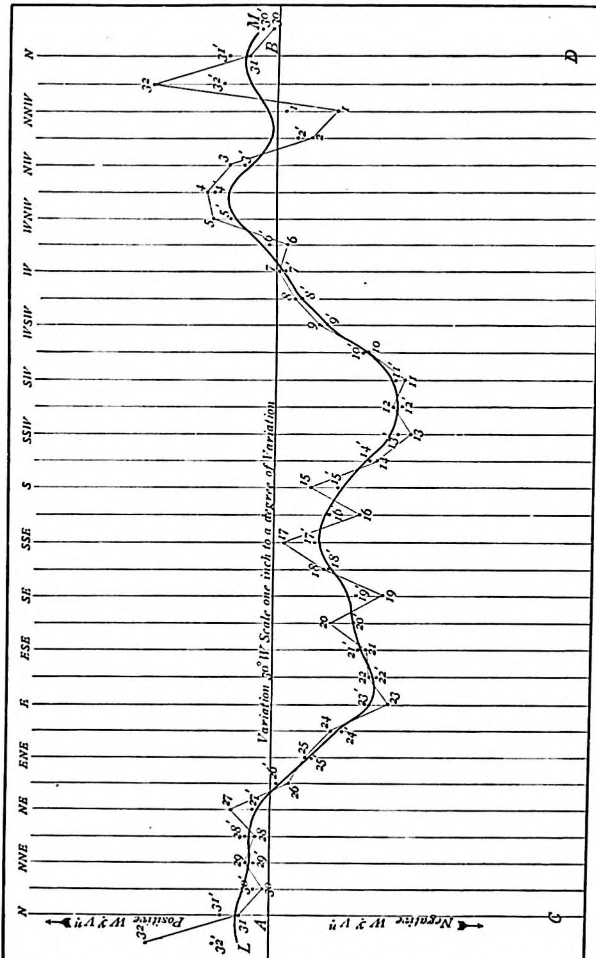

N
NNW
NW
WNW
W
WSW
SW
SSW
S
SSE
SE
ESE
E
ENE
NE
NNE
N
Variation 30° W. Scale one inch to a degree of Variation
Positive
Negative
A
B
L
M
C
D

FIG. 64.

turbing forces would lead us to expect. We, therefore, bisect the straight line joining 1 and 2 and all the others in succession, as well as that joining the points 32 and 1, and join the points of bisection as they follow in order, two and two, with straight lines, cutting the *NW* by *N* blue straight line in the point 2′, the *NW* blue straight line in 3′, and so on. The points 1′, 2′, 3′, ... 31′, 32′, will represent the observations corrected by comparing each of them respectively with the two observations between which it lies, viz., the one immediately preceding it and that which immediately follows it. Again join the points 1′ and 2′, 2′ and 3′, and so on, with straight pencil lines, which will give the chords of another curve, approximating nearer to the true one than the preceding. Where the changes in the directions of the second system of chords change too abruptly the process must be continued until we obtain a series of chords in which the change in the direction in passing from one point to the next is gradual, so that the observations may be each corrected so as to agree, as nearly as possible, with all the others, and each have its fair and proper effect on the curve showing the variations of the compass. The curved line of gradual curvature *LM* is then drawn to pass as nearly as possible through the points, taking the mean direction of the two chords which meet these, and will represent the most probable value of the variations of the compass that can be derived from the observations. The ordinates of this curve being measured and applied to variation 30° W., represented by the axis *AC*, will give the variations recorded in Table VIII.

Bright stars, when of small altitude and moving slowly in true bearing, can be used for determining the variations of the ship's compass in the same manner as the sun, but the latter is preferable.

Table VIII.

Ship's Head by Standard Compass.	Variation of Compass given by Curve.	Difference between the Curve and Observation.	Remarks.
	°	°	
North.	30·4 W.	−0·1	From the Curve we find— Mean variation, - 29·6 W. Greatest, - - 30·6 Least, - - - 28·5 The mean of the differences between the curve and the observation values of the variation is $-\frac{1}{30}$ degrees, the mean size of the differences 0°·23, and the greatest difference 1°·1. The differences in the two last lines of the table are larger than we should expect, but still, as no remark was made against the observations at the time, they ought not to be discarded, but, had time allowed, they should have been repeated.
N. by E.	30·3	−0·2	
N.N.E.	30·3	0·0	
N.E. by N.	30·3	0·0	
N.E.	30·2	+0·3	
N.E. by E.	29·9	−0·1	
E.N.E.	29·6	0·0	
E. by N.	29·2	0·0	
East.	28·8	−0·2	
E. by S.	28·8	0·0	
E.S.E.	29·0	−0·1	
S.E. by E.	29·1	+0·3	
S.E.	29·2	−0·4	
S.E. by S.	29·3	+0·1	
S.S.E.	29·5	+0·4	
S. by E.	29·4	−0·4	
South.	29·2	+0·4	
S. by W.	28·9	−0·1	
S.S.W.	28·6	−0·2	
S.W. by S.	28·5	−0·0	
S.W.	29·6	−0·2	
S.W. by W.	29·0	−0·1	
W.S.W.	29·5	+0·1	
W. by S.	29·8	+0·1	
West.	29·9	+0·1	
W. by N.	30·2	−0·3	
W.N.W.	30·6	+0·2	
N.W. by W.	30·6	+0·3	
N.W.	30·4	+0·2	
N.W. by N.	30·1	−0·5	
N.N.W.	30·1	−0·8	
N. by W.	30·4	+1·1	

CHAPTER XIII.

ON SOUNDING.

BEFORE describing how to sound in a boat it will be convenient to enumerate the instruments and implements which are required for the purpose. They are the following.

2 sextants having clear reflectors and good telescopes, with large field glasses; one must be capable of measuring a small angle accurately on and off the arc.

1 Raper, an instrument invented by a naval officer of that name, in which two reflectors are set in fixed positions in a frame so as to measure an angle of exactly 180°.

2 small station pointers, which lay off the small angle on opposite sides, with high centering, so that near objects may not be obscured by it.

Several squares of tracing paper, in a small case, to be used when the near objects are too close to the centre of the station pointer.

1 watch set to mean time.

1 straight-edged protractor, six inch, with scales of equal parts on it.

1 pair of dividers.

1 straight-edged ruler of about 18 inches.

1 measuring tape.

1 telescope.

1 boat's compass and 1 prismatic compass.

1 pole, with ten feet accurately and distinctly marked upon it.

1 penknife, with pencils and india rubber.

1 angle book and 1 sounding book, each with the boat's name marked distinctly on the cover.

Several flags for marks, each of two colours, which should be contrasts.

Some canvas or thin light boards for sounding marks.

1 bag of unslacked lime, 1 bucket, and brush for whitewashing.

2 lead lines, each about 33 fathoms long, marked to 30 fathoms in the same way as an ordinary ship's line, but with

an additional fathom mark at each *deep*, and also a mark at each quarter of a fathom as far as five fathoms.

2 leads, with tallow for arming them.

Before marking, the lead lines should be well stretched, and the marks placed so that they may be at their proper distances from each other when the line is extended wet along the deck, at a tension of about seven pounds; the length of the lead and its becket is allowed off the first fathom. When used for sounding the lead line must be carefully measured both before and after the operation; its errors, if any, must be noted in the sounding book, because the marks should *not* be shifted for small errors.

The boat should be as light as possible in her build and equipment, in order that her crew may be able to beach her in bad weather; fitted to pull an odd number of oars, at least five, the odd oar being for the leadsman when not sounding; the boat's head sheets must be of sufficient size for the leadsman to stand firmly upon a platform fitted there, with the lead line coiled clear upon it before him; the oars should be light, handy, and well balanced; lug sails, with light masts, *without any gear*, only the halyards; a sailing thwart for strong breezes. The stern sheets of the boat should be able to accommodate two sitters with the sounding sheet stretched on a board before them, and a covered place right across the stern of the boat behind them, with a shelf on which to place the instruments when not in use, and where they can be kept dry.

The sheet of points having been prepared and stretched on a board in the manner previously described in the chapter on stretching, the surveyor should consider carefully and lay down the work he intends doing before starting in the boat; he should so arrange that some work may be continuously done as the boat moves along her course, that she may always be in the right place at the proper time, as regards tide currents, the position of the sun with respect to the lie of the shore, so that he may be able to distinguish the objects he requires to fix the position of the boat, and other things of a general and special nature too numerous to mention, but which soon force themselves into notice, and at times inconveniently so, unless taken into consideration and provided against beforehand.

When the tide range is large the soundings should generally be taken after half-ebb until half-flood, when sounding should cease; the low-line should be sketched at a low water, commencing about one hour before low water and working as long as the tide permits; on these occasions a boat-hook, graduated to feet, will be found useful to point out where the line of low water is when the reading of the tide pole is zero.

The surveyor should rule the lines of soundings he proposes to run on the board lightly in pencil, and should adhere to them as near as he possibly can; the sounding lines, when possible, should be arranged in equidistant parallel lines perpendicular to the mean direction of the green or coast line, in pairs of alternating long and short lines, the long lines extending from the shore to a position where the sounding *at low water* is about 12 fathoms, and the short lines from the shore to 6 fathoms.

The distance between the sounding lines will depend upon the nature of the coast and the scale of projection; the distance between the soundings should be proportional to the depth of water, but between every two fixed positions of the boat they must be equidistant from each other. No special rule can be given, but the surveyor must satisfy himself that no peculiarity in the bottom has escaped his notice, and that he has delineated it accurately on the sounding sheet.

Equidistant parallel lines have the advantage of more thoroughly examining a piece of ground with the same length of sounding lines than if the lines are disposed in any other way.

The word "fix" is employed by surveyors to denote that a position is to be or has been determined by observations; when the observations are made entirely at the surveyor's place it is said to be "fixed independently." This is the method generally adopted when sounding in a boat: two angles are observed at the boat between three known objects properly situated with respect to each other and the boat; the selection of these objects is very important, and the following simple rules will be very useful to guide him in doing so.

1. From the known objects on the sounding sheet select the nearest to the boat, then one that is farthest from the boat, and, thirdly, a near object which makes with the object first selected an angle as near to 90° as possible.

2. When the distances of the known objects from the boat do not differ much from each other, select them so that whilst the boat is moving one of the two angles may change very fast, whilst the other changes very slowly.

3. When the middle point of the three is nearer to the boat than the other two, or when the three points are in the same straight line, or when the apex of the triangle, formed by the straight lines joining the three points, is towards the boat, or when the boat is within that triangle, the projection of the boat will always be possible, but may not be very good.*

* See Shortland *On Sounding*, published by Potter, 1858.

The boat's name should be written on the back of her angle and sounding books, which ought each to have a pencil attached to it with a string.

The sounding book should contain a clear record of all the boat's sounding operations, noted immediately they occur, and the system of notation should be uniform throughout the survey. The day and date should be noted at the commencement of each day's work, with the surveyor's and leadsman's name, the lead line corrections, and those of the instruments determined before the boat started to sound.

When noting an angle he has observed, the surveyor should describe the place by means of a symbol, or by time; then

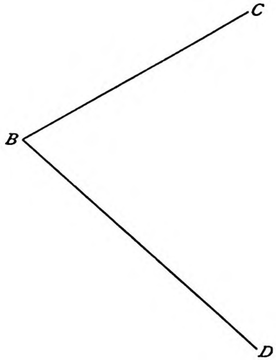


Fig. 65.

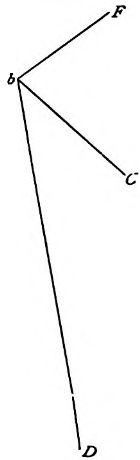


Fig. 66.

write the symbol denoting the left hand object, or describe it, then the number of degrees, etc., in the angle, or the reading of his sextant, and lastly the symbol denoting the right hand object, or describe it; all being written following each other in order on the same line.

Suppose an observer at B, Fig. 65, takes with his sextant the angle CBD, and finds its reading to be $50° 22'$, he notes it thus, the time by his watch being $10^h 12^m$ A.M.,

$$AbB\{C\ 50° 22'\ D\}. \quad \text{Time } 10^h 12^m \text{ A.M.}$$

Suppose next that at $11^h 15^m$ A.M., to fix the position of the

boat b, Fig. 66, the observer takes the angles FbD, CbD, and the readings of his sextant gives angle $FbD = 112^\circ\ 15'$; angle $CbD = 28^\circ\ 45'$, the sounding being $7\frac{1}{2}$ fathoms mud, it should be noted

$$\text{Mud } 7\tfrac{1}{2} \begin{Bmatrix} F\ 112^\circ\ 15'\ D \\ C\ \ \ 28\ \ 45'\ D \end{Bmatrix}. \quad \text{Time } 11^{\text{h}}\ 15^{\text{m}}\ \text{A.M.}$$

The soundings, as soon as the leadsman calls them, should be written in pencil in following order on a line, with a vacant line below on which the corrected soundings are afterwards written in ink under their originals; when the bottom is called it is immediately written over the depth to which it corresponds; the time of the first and last sounding on each page, as well as the time of each fixing, should all be noted, especially when the tide reduction is changing rapidly. After the boat has returned from sounding, the tide-pole readings must be taken from the tide book and written in ink under each time noted during the sounding operations.

The distance which the lines of soundings are to be from each other having been settled, they should be drawn on the sounding sheet, perpendicular to the average direction of the green line and passing through as many of the fixed objects on the coast line as possible, accommodating the distances between the lines in order to do so, it not being necessary to adhere rigidly to the lines when sounding, but to keep as near to them as possible. In placing the running marks at the shore extremities of the lines to be sounded, the ten feet pole can be usefully employed as follows. Suppose the scale of the sheet to be 4 inches to a mile and we wish to place the lines of sounding about half-an-inch distant from each other on the sheet, or about 750 feet from each other on the water; at this distance the 10 feet on the pole will subtend an angle of about 46′, we therefore adopt this angle to define with the pole the distance between two consecutive running marks, when the line joining them is parallel to the mean direction of the shore; but, as the green line on which the sounding marks are placed deflects in places from its mean direction, in such places the distance between two consecutive marks must be increased and the pole angle diminished in proportion to the cosine of the angle of deflection. A common traverse table may be made use of to determine the proper angle: enter the table with the angle of deflection as a course and take out the difference of latitude corresponding to distance 46, the result will be the pole angle required. In this manner the following table was quickly constructed, to be written on a slip of paper and pasted on the

frame of the sounding board, or it may be written in first page of the sounding book.

Angle of Deflection.	Angle subtended by 10 feet.	Angle of Deflection.	Angle subtended by 10 feet.	Angle of Deflection.	Angle subtended by 10 feet.
0°	46′·0	15°	44′·4	30	39′·8
5	45 ·0	20	43 ·2	35	37 ·7
10	45 ·3	25	41 ·7	40	35 ·2

Let *AbcdmB*, Fig. 67, be the green line of the sounding sheet,

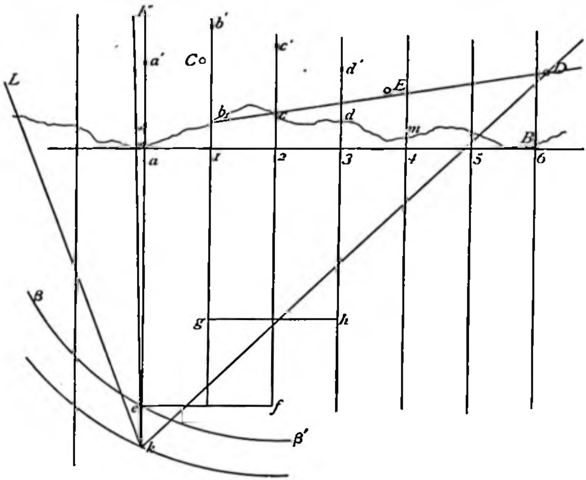


FIG. 67.

A, B, D, E, C, K, L, fixed points on the sheet. Take *AB*, which we will suppose to be the nearest fixed line to the mean direction of the coast line, draw the straight line *AB* in pencil and extend it right across the sheet, take the distance the lines of soundings are to be from each other with a pair of compasses, lay them off from *A* on *AB*, and let 1, 2, 3, etc., be the points thus defined; through these points draw straight lines in pencil perpendicular to *AB* and also through *A* the straight line *eAaa′* perpendicular to *AB*, and suppose *A*1 to be the distance on the sheet corresponding to the distance at which

10 feet vertical subtends an angle of 46′. The board thus prepared is ready to commence work with.

Upon arriving at *A*, the fixed point nearest to the place from which the boat started, if the land inshore of *A* is higher than it, so as to be seen over *A* from seaward, an object, *a′*, such as the chimney of a house, a remarkable tree, a distant peak, or some other well-defined object in the straight line *eAa′* is found and selected for a running mark in conjunction with *A*. To find the running mark set the index of a sextant to 90°, and clamp it securely; then if *B* is to the right of *A* when looking inshore, as in the figure, direct the line of sight of the sextant inshore. The reflected image of *B* will then pass over objects in the straight line *Aa′* and if one be found there sufficiently distinct, choose it for the running mark, and the greater the distance it is from the observer the better. Should no such object be found, a temporary mark must be placed there. Should *B* be to the left of *A* when the observer looks inshore, he must direct the line of sight of the sextant at *B*. The reflected objects in the line *Aa′* will then pass over *B*, and will thus be found. If the land at *A* is too high for the objects inshore of it to be seen, a temporary mark must be placed on the line *Aae*, as near to the edge of the water as possible. The pole bearer is either sent along the green line towards *b*, or is left at *A*, according to the position of the sun, so that the observer may see the marks on the pole distinctly. The angle *bAB* is taken with a sextant, or determined by a prismatic compass, from the difference between the bearing of *B* and *b* when taken from *A*, suppose $bAB = 12°$. Looking in the foregoing deflection table we find the pole angle corresponding to the proper distance of *b* from *A* to be 45′. The pole bearer or the observer, as the case may be, walks along the line *Ab* until the ten feet subtends an angle of 45′. He will then be in the proper position for *b*, where a mark must be placed, and here the observer takes two angles to fix *b*'s position independently, which he notes in his sounding book, together with the angle of deflection measured at *A*, and the angle subdivided by the ten feet on the pole. The observer projects *b* on his board, and whilst walking from *A* to *b* sketches in the green and high water lines, measuring the height of the cliff and sketching in the beach. The observer then determines *b*'s running mark as follows. He selects a distant well-defined known object *D* on his sheet, joins *bD* with a straight pencil line, and with a protractor measures the angle *b′bD*, which he sets on his sextant, and finds *b′* in a manner similar to that by which *a′* was determined from *A*. He then proceeds along the green line to *c*, and afterwards to *d*, determining their

positions and those of the running marks c' and d' in a similar manner, besides sketching in the green and high lines and carefully noting all the observations he makes in the sounding book. The boat then sounds along the straight lines Ae, ef, fc, bg, gh, and hd, commencing as close to the shore on the straight line Ae as possible, keeping A and a' in one when a' is visible, or otherwise by keeping A and a, which will be close to the boat, in line. If the straight line Ae is well defined, one angle between two known objects on the sheet, one on each side of A, will be sufficient to determine the position of the boat at starting; otherwise two angles must be taken between three known objects selected in accordance with Rule 1. The angles, sounding, and time must be noted in the sounding book, the subsequent soundings being noted in the order in which they are taken, with the nature of the bottom written over each depth. Between two consecutive fixings of the boat the soundings should be equidistant from each other, the distance between the soundings being regulated by the depth of water and the scale of the sheet. A good practical way to ensure this is to pull an even stroke, count the strokes, and when a given number of strokes have been pulled lie on the oars. When the boat's way is sufficiently reduced to ensure the leadsman obtaining a good up and down sounding, at the expected depth the lead is hove, and after the leadsman has called the depth the oarsmen commence pulling again, and so on. Three strokes per fathom is a fair allowance for the four inch scale in fine weather and smooth water, which is the sort of weather to be chosen for boat sounding. The stroke oarsman should count the strokes, give the orders to cease pulling and to heave the lead, as well as when to pull again, the two after oarsmen keeping the boat on the course defined by the running marks. When the depth of the water changes gradually, an odd number of soundings, such as 3, 5, 7, 9, etc., should be taken between each two fixed positions of the boat, the more rapid the change the fewer the number; but upon any abrupt change in the depth the boat's position should be immediately determined, as well as when the nature of the bottom changes from mud to rock, or *vice versa*. Suppose when the boat arrives at e the sounding when reduced to low water exceeds 12 fathoms, the position of the boat must be carefully determined by taking two angles between three well selected objects marked on the sounding sheet and projected on the spot. The course of the boat should be altered to pull along the line ef as nearly parallel to AB as possible. This is best done by pulling either from or towards a well-defined object on shore. When

this is not available, a compass course will be sufficient, or to steer with the extreme point of land 5° or 10°, etc., on the bow, remembering that *ef* is a short line, and if not well defined by a running mark the boat's position requires to be determined independently more frequently. Upon reaching the point *f* on the straight line *c′cf*, which will be known by the objects *c* and *c′* coming in one, the boat's position *f* must be fixed independently with care, and the boat's course altered to pull along the line *fc* by keeping *c′* on with *c*. The sounding at *f* having first been taken and noted, the line is sounded up to the shore in the manner already described. The line *bg* is then sounded off shore until the boat arrives at *g*, where the reduced depth first exceeds six fathoms. *g* is fixed and the boat's course altered to pull and sound along the lines *gh* and *hd* in the same way as the longer lines *Ae* and *fc* were sounded. Four new equidistant parallel lines are then marked off by shifting the marks already placed into new positions, or by making new ones, and in this manner the coast should be sounded until the whole sheet is full.

If a closed harbour, which is not too broad nor too deep, is to be sounded, equidistant parallel lines of sounding should be carried right across it from shore to shore. In this case the sounding marks are placed on the shore ends of each line to be sounded, and the boat is kept pulling in a line between the marks by using a Raper. In other respects, as to fixing the boat's position, the same precautions must be used as before described.

It is not always convenient to place sounding marks along the green line in the manner just described, and sometimes it is not practicable. In such cases, having ruled the lines proposed to be sounded on the sounding sheet, the boat picks up one of the off shore points, such as *e* in the straight line *Ae*, in the following manner. Place the centre of a station pointer at *e*, with the edge of its zero leg passing through one of the fixed points on the board near the boat, such as *A*; move its left leg so that its edge may pass through another known point, such as *L*, near the boat, making a sufficiently large angle with *A*; and then move the right leg so that its edge may pass through a point on the sounding sheet distant from the boat, such as *D*; take and note the vernier readings, and set the verniers of two sextants to the same readings. When the boat arrives near the point *e*, which will be known by looking at the point *L* with the sextant having its vernier set to the angle *LeA*, to see if the reflected image of *A* is approaching near to *L*, and also at the *A* with the sextant whose vernier is set to the angle *AeD* to see if the reflected image of *D* is getting near to *A*, seen directly;

observe the angles LeA, AeD with them and project the position of the boat, suppose this to be at k; join ke and measure the angle Lke with a protractor, set the sextant to this angle, look at L and see what well-defined object on the shore, such as k, passes over L, steer the boat straight for k, with the angle LeD set on one sextant, and the angle AeD on the other; whilst pulling towards K the reflected image of D will approach that of L, seen directly; and the other observer, with the angle AeD set on his sextant, will, when looking at A, see the reflected image of D approaching it also; and if the boat is pulling exactly on the line kK passing through e, the contacts will be made at the same instant; but as this is not likely to happen, the observer with the angle LeD set on his sextant keeps it unaltered, and when the points approach sufficiently near he orders the men to cease pulling, and stops the boat the instant the contact is made on his sextant, calling stop to the other observer, who alters his angle if necessary, which, however, is always very slight with able surveyors. At the same instant the lead is hove, and an object in one with A is selected for the running mark, remembering the farther it is from A the better. It is not necessary to stick rigidly to the point e as long as the boat has more than twelve fathoms and she is on or very near the straight line Ae. The sounding may be taken, a running mark selected, and the boat pulled straight for A. Another and very good way of picking up e is to project the boat's position on the board and determine her course for the point e, either by steering from or towards an object on the shore which has been found to be somewhere in the straight line passing through the boat and e, or by taking its compass bearing when no point can be found for a running mark and steering the course by compass. Keep observing the angles at the boat between L and D and A and D, noticing the respective rates at which the two angles change as the boat moves on her course. If the rates of change are proportional to the differences between the observed angles and the angles LeD and AeD respectively, the course of the boat does not require alteration; otherwise the course should be altered slightly so as to bring the rates as nearly proportional to the differences as possible. When the angles observed at the boat approach near to LeD and AeD respectively, change the course of the boat so as to make the angle at the boat between L and D exactly equal to LeD. The boat will then be on the arc $\beta e \beta'$ of the circle passing through L, e, and D. If the angle at the boat between A and D is too small the boat's course must be from L, and so as to keep the angle between L and D its proper size, putting the helm to starboard when it is getting too small and to port when

it is too large. During this time the boat is moving along the circular arc βe, or nearly so; the angle between A and D will increase, and when it becomes equal to AeD the boat will have arrived at e. If, on the contrary, the angle at the boat between A and D is too large, the boat's head must be turned towards L so as to keep it on the starboard bow, and steering the boat so as to keep the angle between L and D equal to LeD. In this way the boat will pull nearly along the circular arc $\beta' e$ towards e; and when the angle between A and D observed at the boat is equal to AeD she will be at e. Great attention should be given to the foregoing, as, with a little practice, it will enable anyone in a vessel or boat to pick up any required position quickly with ease and accuracy. If the land beyond A does not afford a running mark for the boat at e, some rock or stone on the beach between A and the boat must be selected as near to the margin of the water as possible. In this case, the running marks being bad, the position of the boat must be frequently determined independently, and if off the line Ae she must be got on it as soon as possible and another object selected to ensure her being kept on it if possible during the remainder of the sounding on the line. If, however, a good well-defined distant object beyond A has been found, so that the boat can be steered accurately on the line Ae after the point e has been well determined, and the next fixed position of the boat on the line eA towards A has also been accurately fixed, any future position of the boat whilst on the straight line eA can be determined with sufficient accuracy by taking the angle between two near objects, one on each side of the point A. If whilst sounding two pairs of lines such as Ae, fe, bg, and hd, etc., any irregularity in the bottom is found, after fixing the position on the line the boat is sounding, the other lines should be completed, after which the irregularity must be examined if necessary by running short lines parallel to and equidistant from each other between the other lines already sounded, and cutting them across with equidistant parallel lines parallel to the mean direction of the coast line and perpendicular to the others until the shape of the shoal has been found. Its highest parts must then be carefully felt for with the lead and the bottom looked at if the water is clear, so as to ensure the very highest point of the shoal or rock being found and its position determined, as well as to see if good marks on the shore for clearing it can be discovered.

After the day's work in the boat is over, all the soundings must be corrected and reduced to the tide zero, the boat's position carefully reprojected and joined with straight pencil lines, along which the reduced soundings must be written in ink with the nature of the bottom. It frequently happens that the

soundings have been taken too close to enable them all to be written on the sheet, in which case the soundings at the fixed positions must be written in their proper places; then the shoalest and the deepest between two fixed points, and then the others, so that they may be as nearly as possible equidistant from each other. The sounding sheet must then be carefully looked over to see if any part of the day's work requires examination, and this must be all done before the boat proceeds on another day's sounding.

When sounding two stationary objects should always be kept in one. Of these, one should be as near and the other as far as possible from the observer. At the beginning and end of each line of sounding the position of the boat must always be well determined. The two fixed points at its extremities determine the position of the line on the sheet; and the two objects being in one when the sounding is taken ensures its being on the line.

The most distant of the three objects between which the angles are taken to determine the boat's position should, when possible, be the common object observed in both the angles. The nearest, except in a case of necessity, should never be made the common point.

There are three modes of projecting the observer's position from two angles taken between three known objects.

1. The circular projection.
2. The straight line projection.
3. The station pointer, or tracing paper projection.

A clear conception of these projections, the errors they are liable to, and the relative situations of the observer and the objects observed by him which are best suited to reduce the effects of these errors to their smallest limit, is a great assistance to the practical selection of the points best adapted to determine the position of the boat. We will therefore explain how each projection is made, and how the effects of the errors of projection in displacing the position of the boat on the sounding sheet are calculated, and thence deduce some practical rules for selecting the proper objects for observation.

Let A, B, and C be three fixed points on the sheet; b a boat from which the angles $BbA = O_1$ and $BbC = O_2$ are observed, and of which O_1 is less than 90° and O_2 is greater than 90°. To project the boat's place on the sheet by the circular projection join AB and BC with straight lines which bisect respectively in the points M and N. From M draw the straight line MO_1 perpendicular to AB, and from the point N the straight line NO_2 perpendicular to BC. At the points A and B respectively in the straight line AB make the angles BAO_1 and ABO_1, on the

same side that the boat is, each equal to $90° - O_1$. The straight lines AO_1 and BO_1, if the projections are correctly made, will intersect each other and the straight line MO_1 in the same point O_1, the centre of the circle whose circumference will pass

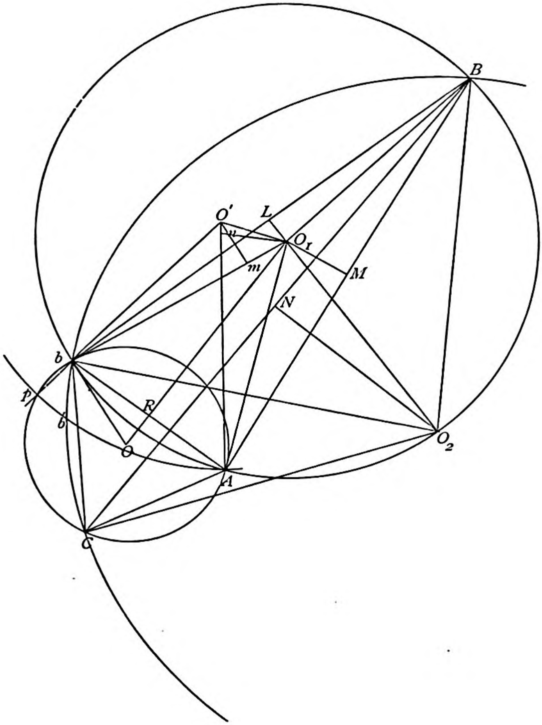


FIG. 68.

through A and B, and of which the segment AbB will contain an angle equal to O_1. At the points B and C in the straight line BC make the angles CBO_2 and BCO_2 on the side opposite to that on which the boat is, each equal to $O_2 - 90°$. The straight lines BO_2 and CO_2 will intersect each other and the straight

line NO_2 in the point O_2, the centre of the circle passing through the points B and C, whose segment BbC contains an angle $= O_2$. It is obvious that the point b in which the circumferences of these circles intersect each other is the position of the boat on the sheet.

The error in the projection of b will arise from the straight lines defining the positions of O_1 and O_2 being erroneously drawn; let O' be the erroneous position of O_1, and $Ab'p$ the circumference of the circle passing through A of which O' is the centre; join O_1O', $O'A$, O_1b, $O'b$, which produce to meet the circumference $Ab'p$ in p.

Let $Ab=d$, $O_1A=r$, $AO_1b=\theta$, $AO_1O'=\phi$, and $O_1O'=\delta$.

Make $An=AO$, and $bm=bO'$; join O_1n and $O'm$; O_1O' is very small when compared with O_1A; we may therefore consider $O'mO_1$ and O_1nO' to be right angles.

$$O_1m=\delta\cos(\phi-\theta), \text{ and}$$
$$O'n=-\delta\cos\phi,$$

and
$$AO'=An+nO'=r-\delta\cos\phi;$$

similarly
$$bO'=bO_1-O_1m=r-\delta\cos(\phi-\theta),$$

but
$$\begin{aligned} bp&=O'p-O'b=O'A-O'b\\ &=\delta\{\cos\overline{\phi-\theta}-\cos\phi\}\\ &=2\delta\sin\frac{\theta}{2}\sin\left(\phi-\frac{\theta}{2}\right). \end{aligned}$$

Let the tangents to the circles AbB and BbC at the point b be inclined to each other at an angle i; then

$$\begin{aligned} bb'&=bp\operatorname{cosec} i\\ &=\frac{\delta d}{r}\sin\left(\phi-\frac{\theta}{2}\right)\operatorname{cosec} i \qquad\text{(1)} \end{aligned}$$

since $\frac{d}{r}=2\sin\frac{\theta}{2}$.

Now O_1AO' is the angular error made in drawing a straight line through the point A making an angle $90°-O_1$ with the straight line AB.

Suppose α to be the average angular error made by the projector with the instruments he uses to draw from a given point in a given straight line a straight line making a given angle with it, combined with the average value of the errors of observation: then the average value of δ under such circum-

stances is $\frac{ra}{\sin\phi}$; substituting this value for δ in (1) we have

$$bb' = da \operatorname{cosec} i \frac{\sin\left(\phi - \frac{\theta}{2}\right)}{\sin\phi}, \qquad \text{.......................(2)}$$

ϕ is unknown, and may have any value: a is a constant with the same projector and observer, using the same instruments, etc., and therefore in order that bb' may be as small as possible, we must make $d \operatorname{cosec} i$ as small as possible, which will happen when d is as small as possible, and $i = 90°$. Therefore the point A must be that nearest to the boat, and the two circles must cut as nearly at right angles as possible.

In like manner the value of the circle of which O_2 is the centre, taken in conjunction with that of which O_1 is the centre, for determining the position of b, may be considered proportional to $bC \times \operatorname{cosec} i$.

Join O_1O_2, which will bisect bB, produced when necessary, in the point L, and join O_2b.

Then $\quad i = O_1bO_2,$

$= bO_1L - bO_2L,$ since bO_1L is the exterior angle of the triangle O_1bO_2.

$= bAB - bCB,$

but $\quad$ angle $bAB = 90° -$ angle $AbB -$ angle bBA,

and $\quad$ angle $bCB = 90° -$ angle $CbB -$ angle bBC,

$\therefore$ angle $bAB -$ angle bCB

$=$ angle $CbB -$ angle $AbB -$ angle $bBA +$ angle bBc

$=$ angle $CbA -$ angle CBA,

$\therefore \quad i =$ angle $CbA -$ angle CBA ;

but when the projection is the best possible, so far as the intersection of the circles goes, $i = 90°$, therefore, for good projection, we must have

angle $CbA -$ angle $CBA = 90°$,

or as near thereto as possible; or

$$O_2 - O_1 = 90° \pm B,$$

where B denotes the angle CBA of the triangle formed by the three points, the positive sign being taken when the point A is on the opposite side of the straight line BC from the point b, where the boat is, and the negative sign when A and b are on the same side of BC.

The two circles passing through C and b may be shown by a similar process of reasoning to intersect each other at right angles in the point b.

When $O_1 = 90° \pm C$, the positive sign being used when the points A and b are on opposite sides of BC, and the negative when on the same side—

Let O be the centre of the circle passing through b, Figure 68, and the points A_1 and C; join O_1O, which will bisect bA in R; the two circles, of which O and O_1 are the centres, will intersect each other at right angles in b when the angle $O_1bO = 90°$;

or when $\quad$ angle bOO_1 + angle $bO_1O = 90°$;

or when $\quad$ angle bCA + angle $bBA = 90°$;

or when $\quad$ angle CbB + angle $BAC = 270°$;

or when

$$\text{angle } CbB = 270° - \text{angle } BAC$$
$$= 90° \pm (B + C);$$

or when

$$O_2 = 90° \pm (B + C),$$

the positive side being taken when A and b are on opposite sides of BC and the negative when A and b are on the same side of it.

Hence when the angle BAC is obtuse, as in this figure, the circles passing through the point A will intersect each other at right angles when

$$O_2 = 90° \pm (B + C);$$

those through point B when

$$O_2 - O_1 = 90° \pm B;$$

those through point C when

$$O_1 = 90° \pm C.$$

When A is the middle point of the triangle ABC as seen from b, B and C its two extreme points, the positive sign being taken when A and B are on opposite sides of BC, and the negative when they are on the same side—

If the point b (Figure 69) be within the triangle ABC, then either two of the three angles AbB, AbC, and BbC having been observed, the third will be given by the condition

$$\text{angle } AbB + \text{angle } AbC + \text{angle } BbC = 360°;$$

and denoting the three angles of the triangle ABC by A, B, and C respectively, when expressed in degrees it may be proved in a manner similar to the foregoing, that when the two circles passing through the point A intersect each other at right angles in the point b, the angle $BbC = 90° + A$; for the two circles passing through the point B to do so we must have angle $AbC = 90° + B$;

when the circles pass through C they will cut each other at right angles in b, when angle $AbB = 90° + C$. These important conditions should be always kept in mind when sounding and practically applied.

Coupled with the above conditions of cutting, the two circles should pass through the point which is nearest to the boat. The circular projection is not used much in practice, especially when station pointers are within reach, but it should be studied carefully in order to understand how to select the best objects with which to determine the observer's position independently.

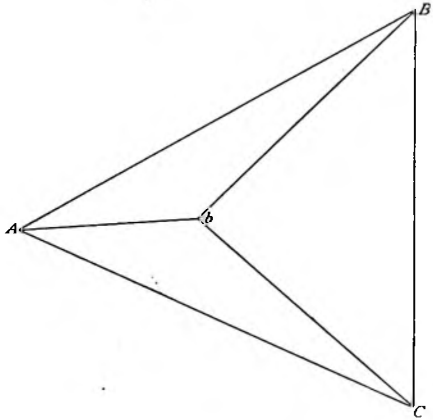


Fig. 69.

Take, for example, the condition of a good projection of b by the two circles passing through the points B, we find that b should be as near to A as possible, and that $O_2 - O_1$ should be as nearly equal to $90° \pm B$ as possible; now, if B is very distant from b, with respect to A and C, the angle B must be very small, therefore $O_2 - O_1$ should be nearly 90°, and we see that when the angle at the boat between two near points is nearly 90°, and the third point is distant from the boat, a good projection will result, which shows the truth of Rule 1.

The straight line projection of the observer's place, from two angles taken between three known objects, is made as follows.

There are two ways in which this projection can be made. First, by drawing the straight line passing through the observer and one of the extreme points of the three given; second, by

drawing the straight line passing through the middle point of the three given and the observer.

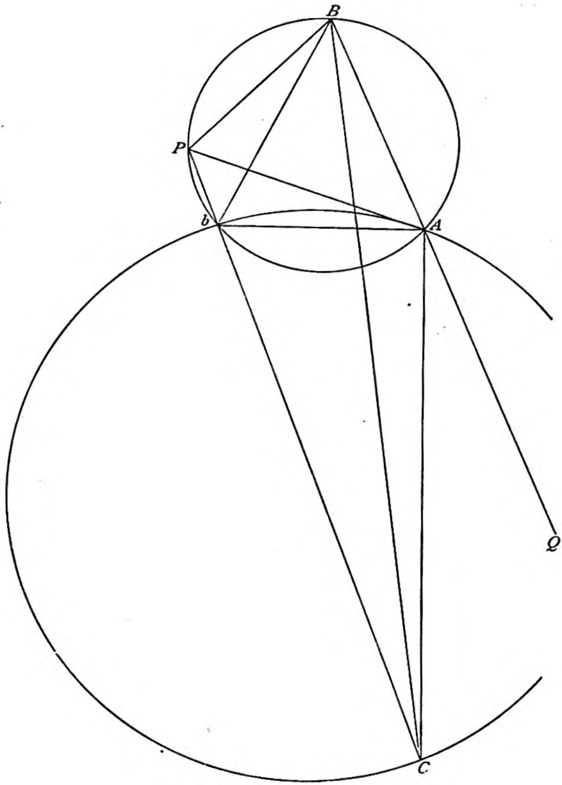


FIG. 70.

A, B, and C, Fig. 70, are three known objects, between which an observer at b takes the angle $BbC = O_1$ and angle $AbC = O_2$.

First, to draw the straight line passing through C, one of the extreme points, and b.

Produce BA to Q, make angle $QAP = O_1$ and angle $ABP = O_2$; let P be the point in which the straight lines AP and BP in-

tersect; join PC and upon AC describe a segment of a circle, AbC, containing an angle equal to O_2, cutting CP, produced if necessary, in b; then b is the observer's place.

Join bA and bB;

$$\text{angle } ABP = \text{angle } AbC, \text{ each being equal to } O_2,$$
$$\text{but angle } AbC + \text{angle } AbP = 180°,$$
$$\therefore \quad \text{angle } ABP + \text{angle } AbP = 180°,$$

Consequently the four points A, B, P, and b are in the circumference of the same circle. Let this circle be described.

$$\begin{aligned} \text{Then angle } AbB &= \text{angle } APB \text{ in the same segment} \\ &= \text{angle } QAP - \text{angle } ABP \\ &= O_1 - O_2; \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{angle } BbC &= \text{angle } AbB + AbC \\ &= O_1. \end{aligned}$$

Therefore b is the position of the observer.

Another very good way of determining the position of b after CP has been drawn, and the one most generally used, is to

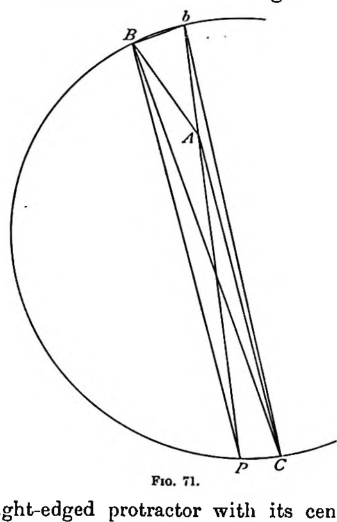


FIG. 71.

place a straight-edged protractor with its centre coinciding with CP and its graduated edge set at an angle O_2 on CP towards C; then slip the protractor thus set along CP, carefully keeping its centre and graduated edge in similar positions

respecting CP, until the ruling edge of the protractor passes through A, the centre of the protractor will then be at the point b; the proof is the same as before, since the angle CbA is by this means made equal to O_2.

To draw the straight line passing through the middle point A of the three given points A, B, C, Fig. 71.

Join BC, make the angle $CBP = O_2$, the angle $BCP = O_1 - O_2$. Join PA and produce it indefinitely towards b.

Upon BC describe the segment of a circle BbC containing an angle equal to O_1, and let it cut PA produced in b; b will be the observer's place.

$$\because \quad \text{angle } BPC + \text{angle } CBP + \text{angle } BCP = 180^\circ$$

$$\text{angle } BPC + O_1 = 180^\circ$$

$$\therefore \quad \text{angle } BPC + \text{angle } BbC = 180^\circ$$

and the point P will be in the circumference of the circle of which BbC is the segment; complete the circle and join bB and bC.

$$\text{Angle } AbC = \text{angle } CBP \text{ in the same segment of circle } PCbB$$
$$= O_2$$

and angle $BbC = O_1$, by construction.

Therefore b is the position of the observer.

The straight-edged protractor can be used to determine the position of b on the straight line PA produced in a manner similar to the previous case, using the angle AbB to the nearer point B.

When the most distant point C is not very far from the observer, the best way of determining the position of b on the straight line PA produced is to lay off the angle O_1 on a piece of tracing paper, place the angular point on the straight line PA produced, so that one of the straight lines containing the angle may pass through the point C and the other may lie in direction towards B, slip the angular point along the straight line PA produced, keeping the straight line passing through C always on that point until the other straight line containing the angle passes through B, the angular point will then be at b; the proof is the same as before.

The accuracy of both these projections depends principally on the accuracy with which the point P can be projected, and its distance from the point through which the straight line defining the position of b is to be drawn being great, viz., we should always have $CP > Cb$ in the one case, and $AP > Ab$ in the other. Referring to Fig. 71, where the straight line passes

through C, the error in the projection of P depending on errors of observation and projection $\propto \frac{AB \sin PAB \,.\, \sin PBA}{\sin^2 APB}$.

Therefore the nearer APB is to a right angle, and the smaller $AB \sin PAB \sin PBA$, the better.

This object will be attained by A and B being as near each other as possible, with the angle AbB as near 90° as possible, and the angle AbC less than 90°, but as near thereto as possible. When these conditions are satisfied, a good projection of b will result from making the straight line on which it lies pass through the point C.

In Fig. 71, where the straight line is drawn passing through the middle point A and the observer, the error in P's place $\propto \frac{BC \sin PBC \,.\, \sin BCP}{\sin^2 BPC}$.

The angle BPC must therefore be as near 90° as possible, the points B and C as near each other as possible, with the angle PBC or PbC very small.

Also the smaller $\frac{Ab}{AP}$ the smaller will the effect of an error in P's place have on that of b.

We should therefore have A nearer to b than either B or C are, with B and C near to each other, and the angle BbC nearly 90°, whilst AbC is very small, when we draw the straight line to pass through A the middle point and the boat b.

I have purposely dwelt on these projections, although in practice they are seldom used, because they are so very important to guide the surveyor to a good selection of points for the purpose of projecting his position independently.

Let A, B, and C (Fig. 72) be three given points on the sounding sheet; S_1 the centre of a station pointer, on which the two observed angles between A, B, and C have been set, and which has been so placed on the sheet that the feather edges of its legs pass respectively through A, B, and C, and coincide with the straight lines S_1A, S_1B, and S_1C.

Suppose the station pointer be moved so that the feather edges of the legs passing through the points A and B may still pass through them, whilst that of the third leg takes the position S_2cM, cutting S_1C in c. S_2 being the new position of the centre of the station pointer, join S_2A and S_2B.

Since angle AS_2B = angle AS_1B; and angle cS_2B = angle cS_1B, the points A, B, C, S_1, and S_2 are all in the circumference of the same circle. Let it be described, and through S_1 draw TS_1T', touching the circle in S_1. From centre c, with distance cC, describe the circular arc CM, cutting S_2cM in M.

The value of this projection may be considered proportional to $\frac{CM}{S_1S_2}$, in other words, the larger CM is with respect to S_1S_2 the better the projection.

$\because$ angle S_1cS_2 = angle CcM,

and angle $S_1cS_2 = \frac{S_1S_2 \sin AS_1B}{AB}$,

also angle $CcM = \frac{CM}{Cc}$;

$$\therefore \quad \frac{CM}{Cc} = \frac{S_1S_2 \sin AS_1B}{AB};$$

$$\therefore \quad \frac{CM}{S_1S_2} = \frac{Cc \sin AS_1B}{AB}$$

$$= \frac{(S_1C - S_1c)\sin AS_1B}{AB}.$$

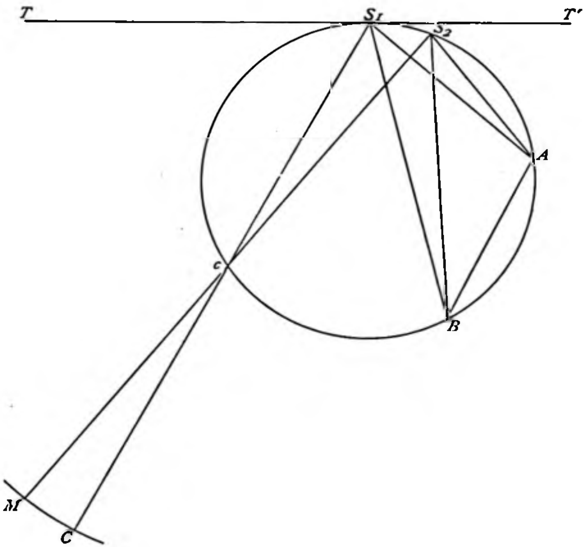


FIG. 72.

Consequently the larger $\frac{(S_1C-S_1c)\sin AS_1B}{AB}$ is the better will be the projection.

First, the farther C is from S_1 the larger S_1C is. Secondly, $S_1c=0$ when the point C is on the tangent S_1T, or is on the opposite side of S_1T from what it is in the figure. This condition will be fulfilled when the observed angle BS_1C is not less than the angle BAS_1. Thirdly, the nearer AS_1B is to a right angle the larger its sine will be; therefore the observed angle AS_1B should be as near a right angle as possible. Fourthly, the smaller AB is the better, or we ought to have the two points A and B as near to each other as possible.

The following simple rules derived from the straight line and station pointer projections, taken in conjunction with the conditions regarding the relations between the observed angles and those of the triangle ABC deduced from the circular projection, will ensure a good selection of points for observation.

Rule 1. The known object nearest to the observer must always be taken.

Rule 2. A near object, which makes at the observer an angle as near 90° as possible with the object first selected, should next be fixed on.

Rule 3. A distant object must then be found. This should if possible be on the *opposite* side of the first selected object from that on which the second is.

Rule 4. When the distant object is of necessity on the same side of the nearest as the second selected known object is, the angle at the observer between it and the second object should not be less than the angle at the first selected point between the second selected object and the observer.

Having, in the manner before described, carried the boat standing off shore as far as a depth of not less than twelve fathoms at low water, and so that the outer sounding of each of the long lines may lie as nearly as possible in a straight line parallel to the mean direction of the coast line off which they are situated, we must turn our attention to the sounding outside these lines. These may be divided into two sets.

First, the soundings which are sufficiently near the shore to be fixed independently by angles between the known objects on it. Secondly, the soundings which, for the most part, are so far off shore that they cannot be fixed in that manner.

For the first set a small steamer or, where none is available, a fore and aft vessel of from 50 to 100 tons will answer the purpose. If a steamer is employed, she should run lines of

sounding forward and back along and parallel to the mean direction of the coast line, the first line being distant from the lines of the outer boat sounding about $\frac{4}{3}$ of the distance between the lines of sounding run off shore in the boats, the next line outside this about $\frac{5}{3}$ the distance from the first, and so on, the distance between the lines increasing with their distance from the shore. This is only a general description, and supposes the depth of the water to increase gradually and not very rapidly as the distance from the shore increases. No precise rule can be given, because the practice must vary with the peculiarities of the place, and must be left to the judgment of the surveyor.

The lines of sounding intended to be run by the steamer should be drawn lightly on the sheet with pencil, the angles defining the position of the first sounding being taken from the chart as well as the magnetic direction of the line of sounding. The vessel is brought into and stopped at the desired position or as near to it as possible in the same way as before described, the lead hove, the angles and time taken and noted in the sounding book, and the vessel's position projected on the sounding sheet. The depth having been found, the engines are started ahead at a given number of revolutions, the line hauled in and the vessel steered on the course given by the magnetic direction of the line drawn on the sheet. When the lead is in, the depth and bottom must be noted in the sounding book. When the vessel has run on the course the time arranged to give a proper distance between the soundings, the engines should be eased, or eased and stopped if necessary to ensure a good sounding being obtained, the times of easing and stopping being noted in the sounding book, two angles taken when the lead is hove, and the vessel's position projected; this will show whether the course and distance between the leads has been such as was desired. If not, the course, number of revolutions of the engine, etc., must be altered so as to produce the desired result. The next lead is hove and the vessel fixed in the same manner, and so on until the vessel's course and the distance between two soundings has been brought to that previously determined—when the leads can be hove by time, continuing the same course so long as the surveyor is satisfied the vessel is sufficiently near to the places he wishes the lead to be hove. The number of revolutions, the intervals of full speed, easing, and stopping must be all kept exactly equal, and the course carefully steered when the soundings between two fixed positions of the vessel may be considered to be on the straight line joining them and equidistant from each other.

When a fore and aft vessel is to be used, and she has a soldier's wind for running along shore, she can run the lines in a manner

similar to a steamer, though her position will require to be more frequently fixed; with the wind blowing along shore the vessel runs the line along shore with the wind and then beats back, making her tacks so that the lines of the starboard and port tacks may be respectively parallel to and equidistant from each other, care being taken to heave at equal intervals of time from each other, and under exactly similar circumstances, so that the soundings may be disposed evenly over the sheet equidistant from each other. If well handled and attended to, a fore and aft vessel in moderate weather will do this sort of sounding very well and quickly so long as the soundings do not exceed from 50 to 70 fathoms; the common deep sea line and lead with Burt's buoy and nipper, or Massey's patent sounding machine, will accomplish the work satisfactorily. When using the former the lines must be carefully measured before and after sounding, and when the line is running out the part between the vessel and the buoy must be kept slack and carefully attended, so as not to draw the buoy towards the vessel.

Massey's machines must be tried before and after they are used, and their corrections ascertained as follows. When the vessel is in still water, where a good up and down sounding with the line can be obtained, the lead with Massey's patent machine set at zero is hove in the usual manner, the line hauled taut up and down from the lead, and marked securely with a piece of twine at the surface of the water, the line is then hauled in, the machine read, and the lead rehove without altering the index, the line hauled taut and re-marked with twine as before, hauled in, machine read, and the process continued until the machine reads about 100 fathoms. The line is then carefully measured and the depth of the machine from the surface of the water corresponding to each reading ascertained and noted; adding these together and comparing it with the last reading of the machine, we shall obtain the average correction due to 100 fathoms.

Whenever a good up and down sounding can be obtained the line should be marked and carefully measured, and the depth by line noted with the machine reading. When using Massey's machine it is advisable to use two and heave them alternately.

The off-shore soundings outside those already described can only be well and systematically obtained from a *fast steamer* working during fine clear weather, which is essentially necessary.

A system of parallel lines perpendicular to the mean direction of the coast line and about four miles distant from each other should be drawn lightly on the reduced Mercatorial sheets, the inner ends of the lines being sufficiently near the shore to en-

able the vessel to pick them up by means of the known objects on it.

Stations on high points of land, as near to the shore as possible, and from 15 to 20 miles distant from each other, should be selected along the coast; where observers with good theodolites, and chronometers accurately compared with each other and with those on board the sounding vessel, should be posted.

The observers at these stations cut off the steamer's main truck simultaneously at stated times whilst she is running the lines of sounding.

The deviation of the steamer's compass, as well as the variation of the compass, must be very carefully determined both before and after the lines of sounding are run; whenever practicable whilst sounding the *true* course steered by the ship should be determined by observation.

Three courses should alone be steered.

1. The off-shore course perpendicular to the mean direction of the coast line.

2. The course parallel to the mean direction of the shore which joins the off-shore ends of a pair of sounding lines.

3. The on-shore course perpendicular to the mean direction of the shore.

The lines should be run in pairs about eight miles apart, alternately long and short, their lengths depending on the speed of the vessel, a short pair taking about eight hours and a long pair about sixteen hours to run; the lines of each pair being about eight miles distant from each other, with one long line half way between a pair of short lines, or about four miles distant from each, the soundings on the shore lines, and those on the long lines as far off-shore as the ends of the short lines should be taken about four miles distant from each other, and those on the long lines beyond or further off-shore than the ends of the short lines about eight miles distant from each other.

Two patent logs, such as Massey's, should be used for measuring the distances run through the water by the steamer, one on each quarter towed from the end of the spars projecting full 20 feet beyond the vessel's side.

Patent sounding machines, such as Massey's, of which the corrections have been carefully ascertained, should be kept ready for use, and two at least hove consecutively. Their corrections must always be determined after the line of sounding has been finished as well as whenever a good up and down sounding with the line can be obtained.

The steamer should take up her position at the inner end of the line intended to be run off-shore by means of two given angles between three known points on the shore, which are

distinctly visible from her, with her engines stopped, the vessel nearly stationary heading on her intended course, at the time arranged for the sounding to be taken; the signal must be given at the *exact* time ordered, and the lead hove. The instant the observers on the shore see the ball drop from the vessel's main truck, they cut it off with their theodolites, noting the time shown by their chronometers; when the lead strikes the bottom, the engines must be started ahead, at the given number of revolutions before arranged, which should be carefully maintained by the engineers throughout each pair of lines. The patent logs, set at zero, must be dropped into the water the moment the vessel moves ahead on her course, the lead line hauled in, the machine read, bottom examined, and every particular such as angles, time, depth, bottom, course steered, number of revolutions of the engines, patent logs, etc., noted accurately in the ship's deck-book. The exact time at which each lead is to be hove having been pre-arranged with the observers on shore must be accurately kept; the intervals of time from full speed after easing the engines to stopping them must be kept the same throughout each pair of lines, and must be arranged to suit the deepest sounding likely to be found, so that the motion of the vessel during each stopping time for sounding may be as nearly as possible the same for all the soundings. When the appointed time for the second sounding is near at hand the engines are eased to a given number of revolutions for a given time and then stopped for a given time, so that at the exact instant at which the second lead is to be hove the vessel may have very nearly lost her way through the water—at this instant the signal must be given and the lead hove, angles taken between three known objects on the shore if visible; if only one or two are visible, and the sun is above the horizon, it must be used for the distant point in conjunction with them. Once an hour whilst the vessel is stopped the altitude of the sun and the chronometer time should be accurately observed, together with the compass bearing of the sun and the direction of the ship's head, which should be very nearly if not exactly on her course. At each time of stopping the patent logs must be carefully hauled in, read, and replaced in the water.

If a small vessel is available and the depth not too great she should be anchored as far off shore as possible on a line parallel to the sounding lines which passes about half way between the two shore observers, but so as to be seen by both in order that they may determine her position. From this vessel observations should be made hourly on the current, wind, drift of the sea, noting when the steamer is in sight,

when she stops for sounding, taking a line of bearing to her, the times being accurately noted, as well as the depth of water in which she is anchored, and the nature of the bottom, with a sketch of the appearance of the land.

If the currents are regular, and known to some extent, a small allowance must be made on the steamer's courses, so as to bring the off-shore and on-shore lines as near their intended directions as possible.

The observers at the two stations should cut off the steamer's main truck *exactly* one minute before the given time, at the given time, taking it from the ship's signal if seen distinctly, and noting the time by their chronometers, and exactly one minute after the given time, carefully reading and noting the A-verniers of their theodolites as well as their chronometer times. The advantage of taking the three readings of each theodolite is that if any small discrepancy in the time should cause the two middle readings not to be exactly simultaneous, one theodolite reading may be corrected so as to make it simultaneous with the other, at the instant the lead was hove.

Whilst sounding, the chronometers should be compared every evening by means of rockets fired from the steamers; besides, during the day, when the ship's signal is seen distinctly the instant the ball and flag are dipped affords a means of comparing the chronometers.

From the theodolite readings the true bearing of the steamers from each station is determined. To each true bearing thus found half the inclination between the steamer's meridian and that of the observer must be applied to find the Mercatorial bearing; the Mercatorial bearings of the vessel, being drawn from each of the two stations respectively on the Mercatorial chart, cut each other in the vessel's position at the time the sounding was taken.

The bearing and distance of two positions of the steamer thus found, when compared with the course and distance run by the steamer between them given by the compass course, corrected for variation and deviation, and the patent logs, afford a means of determining the errors of steerage, the errors of the patent logs, the rate and direction of the current, etc. When a sufficient number of fixed positions of the steamer have been obtained to form at least as many equations as the unknown quantities it is necessary to find, the farther the fixed positions of the ship are from each other the better as a general rule will be the results, except as regards current, when the longer the time and the nearer the two positions are the better in general will be the result.

The following is an example of this method.

On the 8th September, 1864, observations were made at Hospital Hill and Mother's Hill, on the coast of Nova Scotia, to determine the positions of H.M.S. *Buzzard* at given times whilst sounding. Hospital Hill station will hereafter be denoted by the letter H., and Mother's Hill by M.

At 10^h A.M., by observation from H., the true bearing of the *Buzzard's* main truck was S. 34° 12′ W., whilst its true bearing observed from M. at the same instant was N. 73° 39′ E.; the approximate departure between H. and the *Buzzard* was 4′·3, and that between the meridians of M. and the *Buzzard* 22′·2, at the same instant the following angles were observed at the *Buzzard*:—

Street in Island Station 31° 3′ Prospect Church Spire.
Prospect Church Spire 70° 47′ Sambro Light House.
Sounding at ship, 47 fathoms, mud and sand.

Two patent logs, set at zero, were placed over each quarter, and towed from the extremity of spar projecting 20 feet beyond the ship's side; the engines were started ahead, the ship's course corrected for deviation and variation being S. 42° E.

At 10^h 30^m A.M. From H. true bearing *Buzzard*, S. 17° 40′ W.
From M. true bearing *Buzzard*, N. 79° 29′ E.
Departure between *Buzzard* and H. 2′·6.
Departure between *Buzzard* and M. 23′·9.
At *Buzzard*, {Prospect Church Spire, 66° 26′ Sambro Lt. Ho.
{H. 32° 50′ Sambro Lt. Ho.
Distance run by *Buzzard*, mean of patent logs 2′·65, course S. 42° E.
Sounding, 59 fathoms, mud and fine sand.

At 11^h 0^m A.M. From H. true bearing *Buzzard* S. 3° 51′ W.
From M. true bearing of *Buzzard*, N. 86° 22′ E.
Angle observed at *Buzzard*, Prospect Church Spire 52° 19′ Sambro Light House.
Mean reading patent logs 5′·88, course S. 42° E.
Departure between *Buzzard* and H. 0′·7.
Departure between *Buzzard* and M. 25′·8.
Sounding, 63 fathoms, black sand.

At 11^h 30^m A.M. At H. true bearing of *Buzzard*, S. 4° 24′ E.
She was not seen from M.

Angle at *Buzzard*. Prospect Church Spire 38° 49′ Sambro Light House.
Departure between *Buzzard* and H. 1′·12.
Distance by patent logs 9′·0. Course S. 42° E.
Sounding, 71 fathoms, black mud.

At 12h noon. True bearing of *Buzzard* from H. S. 10° 39′ E.
She was not seen from M.
Departure between *Buzzard* and H. 3′·2 approximately.
Distance by patent logs 12′·4. Course S. 42° E.
Sounding, 73 fathoms, dark sand and stones.
Altered course to S. 48° W. Corrected for variation and deviation.
0h 36m P.M. Sounded in 78 fathoms, rock.
Distance by patent logs on course, S. 48° W., 3′·6.

At 1h 0m P.M. At H. true bearing of *Buzzard*, S. 6° 17′ W.
Departure of *Buzzard* from H., 2′·4 approximately.
She was not seen from M.
Distance by patent logs, on course S. 48° W., 6′·1.
Sounding 83 fathoms, dark sand and stones.
Altered course to N. 42° W.

At 1h 30m P.M. At H. true bearing of *Buzzard*, S. 15° 10′ W.
Departure between H. and *Buzzard*, 5′·1 approximately.
She was not seen from M.
Distance run by patent logs on N. 42° W. course, 3′·45.
Sounding, 82 fathoms, rock.

At 2h 0m P.M. From H. true bearing of *Buzzard*, S. 26° 0′ W.
From M. true bearing of *Buzzard*, N. 101° 56′ E.
Buzzard's departure from H., 8′·2 approximately.
Buzzard's departure from M., 18′·3 approximately.
Distance run by patent logs on N. 42° W. course, 7′·0.
Sounded in 77 fathoms, fine sand, with black specks.

At 2h 30m P.M. From H. true bearing of *Buzzard*, S. 38° 21′ W.
From M. true bearing of *Buzzard*, N. 95° 10′ E.
Departure of *Buzzard* from H., 11′·2 approximately.
Departure of *Buzzard* from M., 15′·3 approximately.
Distance run by patent logs on N. 42° W. course, 10′·5.
Sounding, 58 fathoms, sand, with black specks.

At 3h 0m P.M. From H. true bearing of *Buzzard*, S. 48° 51′ W.
From M. true bearing of *Buzzard*, N. 86° 24′ E.

Departure of *Buzzard* from H., 13′·7 approximately.
Departure of *Buzzard* from M., 12′·8 approximately.
Distance run by patent logs on N. 42° W. course, 13′·7.
Sounding, 40 fathoms, sand.

The Mercatorial bearing can be derived from the true bearing in this latitude, with sufficient accuracy for our purpose, by applying half the departure. The number of miles thus given will express the number of minutes of angle at which the Mercatorial meridians are inclined to the true meridians passing through the *Buzzard* and the station from which her Mercatorial bearing is to be laid off. Thus at 10^h A.M. we have

From H. true bearing of *Buzzard*, - - -	S. 34° 12′ W.
Half departure between *Buzzard* and H., -	−2
From H. Mercatorial bearing of *Buzzard*, -	S. 34° 10′ W.

From M., at same time, true bearing of *Buzzard*,	N. 73° 39′ E.
Half departure between *Buzzard* and M., -	+11
From M. Mercatorial bearing of *Buzzard*, -	N. 73° 50′ E.

These two Mercatorial bearings, when laid down on the chart (Fig. 73), cut each other in the point 1. The angles observed on board the *Buzzard* at this time, between street in Island Station, Prospect Church Spire; H., and Sambro Light House, when laid off on a piece of tracing paper, placed on the sheet with the angular point on the point 1, and the left hand straight pencil line passing through or over sheet in Island Station, the other pencil straight lines will be found to pass over the other points in succession pretty accurately, and thus confirm the position of the *Buzzard* at 10^h A.M., as given by the point 1 on the sheet. If this had not been the case the position of 1 would have required a change, so that the observed angles at the ship might have had their fair influence in determining her position. In a similar manner the *Buzzard's* positions at 10^h 30^m, 11^h A.M., 2^h, 2^h 30^m, and 3^h P.M., marked respectively 2, 3, 4, 5, and 6 on the sheet, were determined. At 11^h 30^m A.M., Noon, 0^h 30^m, 1^h, and 1^h 30^m P.M., the *Buzzard* was only observed from H., so that her Mercatorial bearings at these times from H. give only one straight line passing through her, and these were projected in a similar manner, giving the straight line *HcC* passing through

the *Buzzard's* position at 11ʰ 30ᵐ A.M., *HNnag* passing through her at noon, *HPpb*8 passing through her at 1ʰ P.M., and *HdD* passing through her at 1ʰ 30ᵐ P.M.

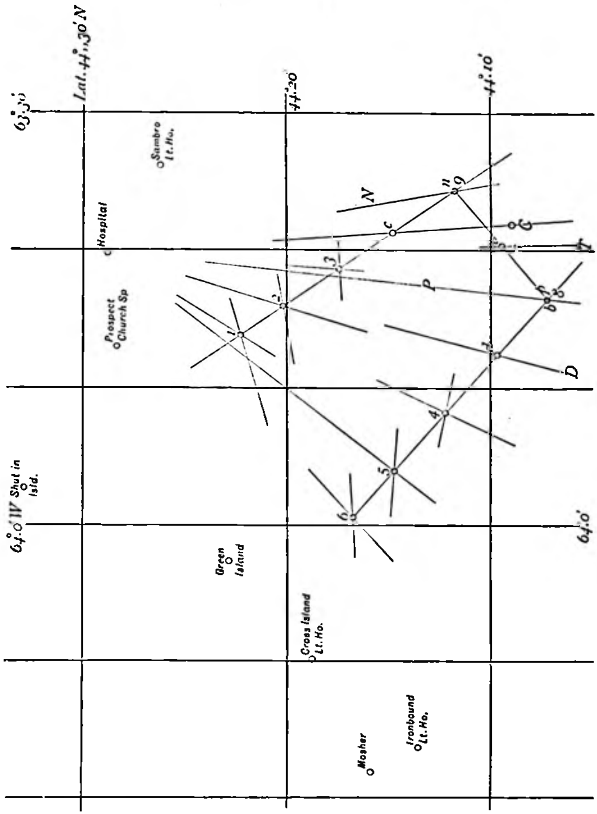


FIG. 73.

Draw a straight line passing as nearly as possible through the points 1, 2, and 3, and so that if they do not all lie exactly on the same straight line, the points 1 and 3 may be on one side and point 2 on the other, the distance of 2 from the

straight line being twice that of 1 and 3 respectively from it, which should be equal to each other. The perpendiculars to this straight line through the points 1, 2, and 3 will give new positions for these points, which must be taken as correct. Producing this straight line until it cuts the straight line $HNnag$ in n, we find a first approximate position of the *Buzzard* at noon. In a similar manner, drawing a straight line as nearly as possible through the points 4, 5, and 6, and correcting their positions if necessary, the point p, in which when produced it cuts the straight line $HPpb8$, will give the *Buzzard's* position at 1^h P.M. to a first approximation.

With a protractor measure the angle which the straight line 1 3 n makes with the meridians; and with a pair of dividers measure the exact length of the part intercepted between 1 and 3. This will give the course and distance made good by the *Buzzard* from 10^h to 11^h A.M. Thus we find that she made during that interval course S. 36° E., distance 6′·1 miles. By dead reckoning we have—course steered during same interval S. 42° E., distance by patent logs 5′·9 miles. In like manner 4 6 gives course made good from 2^h P.M. to 3^h P.M., N. 50° W., distance 7′·3 miles. By dead reckoning during same interval we have—course steered N. 42° W., distance by patent logs 6′·7 miles.

From 11^h A.M. to noon the *Buzzard* steered S. 42° E., and made by patent logs a distance of 6′·5 miles. Supposing her course and distance to be as much affected during this hour as during the one previous to it, we shall have—course made good from 11^h to noon S. 36° E., distance 6′·72 miles. Take this distance off the scale, place one leg of the compasses on the point 3, and with the other describe a small circular arc, cutting $HNnag$ in g. Then g is another approximate position of the *Buzzard* at noon, which may be supposed equally correct to that of n. Therefore bisecting ng we find the point a the most probable position of the *Buzzard* at noon.

In a similar manner we find b the most probable position of the *Buzzard* at 1^h P.M. Join $3a$, ab, and $b4$, which will represent the respective courses made good by the *Buzzard* between 11^h A.M. and noon, noon and 1^h P.M., and from 1^h P.M. to 2^h P.M.

Supposing the current to have been uniform during the interval between 11^h A.M. and 3^h P.M., and that a small error, due to steerage and errors in deviation, affected the S.E. and N.W. courses equally, but in opposite directions, we have sufficient data to determine them as well as the error of the patent log.

For this purpose let r be the rate of the current in miles

per hour; θ the angle which its direction makes with the S. 42° E. and N. 42° W. line on the chart; x the error in the patent log per mile of distance through the water; ϕ the small angular error, due to errors of steerage and of deviation, affecting the S. 42° E. and N. 42° W. courses equally, but in opposite directions.

Hence the course and distance from 10^h to 11^h A.M., represented on the chart by the straight line 1, 3, give

$$5{\cdot}9+5{\cdot}9\times x-r\cos(\theta+\phi)=6{\cdot}1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$r\sin\theta=6{\cdot}1\sin(6^\circ+\phi) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

that from 2^h to 3^h P.M., represented on the chart by the straight line 4, 6, give

$$6{\cdot}7+6{\cdot}7\times x+r\cos(\theta-\phi)=7{\cdot}3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$
$$r\sin\theta=7{\cdot}3\sin(8^\circ-\phi) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

equations (2) and (4) give

$$\frac{\sin(6^\circ+\phi)}{\sin(8^\circ-\phi)}=\frac{7{\cdot}3}{6{\cdot}1};$$

$$\therefore \qquad \tan\phi=\frac{7{\cdot}3\times\sin 8^\circ-6{\cdot}1\times\sin 6^\circ}{7{\cdot}3\times\cos 8^\circ+6{\cdot}1\times\cos 6^\circ}.$$

Calculating to three places of figures in the logarithms in the following manner—

log 7·3	0·863	log 6·1	0·785	log 7·3	0·863	log 6·1	0·785
log sin 8°	9·144	log sin 6°	9·019	log cos 8°	9·996	log cos 6°	9·998
	0·007		1·804		0·859		0·783
log	1·017	log	0·637	log	7·23	log	6·07
	0·637				6·07		
	0·380				13·3		

log 19 - - -	1·279
log 665 - - -	2·823
log tan ϕ - - -	8·456

$$\therefore \qquad \tan\phi=\frac{38}{1330}=\frac{19}{665}$$

and $\qquad \phi=1^\circ{\cdot}38'$

Substituting this value of ϕ in equations (2) and (4) we find

$$\left.\begin{aligned} r\sin\theta&=6{\cdot}1\times\sin(7^\circ\ 38')=0'{\cdot}81\\ r\sin\theta&=7{\cdot}3\times\sin(6^\circ\ 22')=0'{\cdot}81\end{aligned}\right\}\ \ldots\ldots\ldots\ldots \quad (5)$$

By adding (1) and (3)

$$12{\cdot}6\times x+2r\sin\theta\sin(1^\circ\ 38')=0'{\cdot}8$$

or $$6{\cdot}3 \times x + r \sin\theta \sin(1^\circ\ 38') = 0'{\cdot}4$$

$\log r \sin\theta$ - - - - -	$\bar{1}{\cdot}908$
$\log \sin(1^\circ\ 38')$ - - - -	$8{\cdot}455$
$\log 0{\cdot}023$ - - - - -	$\bar{2}{\cdot}363$

$\therefore$ $$6'{\cdot}3 \times x = 0'{\cdot}4 - 0'{\cdot}023 = 0'{\cdot}377\,;$$

$\therefore$ $$x = \frac{377}{6300} = 0{\cdot}06.$$

Subtracting equation (1) from (3)

$$0'{\cdot}8 \times x + 2r\cos\theta \,.\, \cos(1^\circ\ 38') = 0'{\cdot}4$$

or $$0'{\cdot}4 \times x + r\cos\theta \,.\, \cos(1^\circ\ 38') = 0'{\cdot}2\ ;$$

$\therefore$ $$r\cos\theta = \frac{0{\cdot}2 - 0{\cdot}024}{\sec(1^\circ\ 38')} = 0{\cdot}176 \ldots\ldots\ldots\ldots(6)$$

Dividing equation (5) by (6)

$$\tan\theta = \frac{810}{176}$$

$\therefore$ $$\theta = 77^\circ\ 45'$$

log 810 - - -	$2{\cdot}908$
log 176 - - -	$2{\cdot}245$
log tan θ - - -	$10{\cdot}663$

From (5) $$r = \frac{0{\cdot}81}{\operatorname{cosec}(77^\circ\ 45')} = 0{\cdot}83$$

log 0·81 - - -	$\bar{1}{\cdot}908$
log cosec (77° 45′) -	$0{\cdot}010$
log 0·83 - - -	$\bar{1}{\cdot}918$

Direction of current N. $(42^\circ + \theta)$W. or N. 119° 45′ W.

Hence current runs S. 60° 15′ W. 0′·83 miles per hour, $x = 0{\cdot}06$, and therefore distances given by patent log require to be multiplied by 1·06.

The accuracy of the foregoing determinations depends on the assumptions we made being accurate. In this instance we have some means of testing their accuracy as follows.

First, comparing the course and distance made good by the *Buzzard* from noon to 1^h. P.M., represented on the chart by the straight line ab, with her course and distance by dead reckoning corrected thus. Current set the ship, from noon to 1^h P.M., S. 60° W. 0·83. The course by compass, corrected for variation and deviation, during this interval was S. 48° W.; therefore the angle between the ship's course and the direction

of the current was 12°. Taking this angle as a course and 0·83 as distance, we find from an ordinary traverse table that the part of current in direction of ship's course increasing her distance made good, or over the ground, was 0·81, and the part perpendicular to her course setting her more to the westward was 0′·17.

Distance by patent logs was 6′·1.

Multiplying by 1·06 we find distance run by ship through water, 6′·47, set by current during the interval, 0′·81, distance made good, 7′·28. Taking 0·17 as departure and 7′·28 as difference of latitude in a traverse table we find 1° 20′ for the corresponding course; applying this to S. 48° W. we find the course made good by *Buzzard* between noon and 1 P.M., calculated from the corrections, to be S. 49° 20′ W., and the distance made good 7′·28 miles; taking the length and direction of *ab* from the chart we have *Buzzard's* course from noon to 1 P.M., S. 49° W., distance made good 7′·35, a sufficiently near agreement.

Again, at 11^h 30^m A.M., the *Buzzard* was observed from H. to be on the straight line *AcC*, and the point *c* in which the straight line 3*a* cuts it denotes her position at that time, and 3*c* is the distance made good by the *Buzzard* between 11^h to 11^h 30^m A.M.; measuring 3*c* we have 3′·12 for this distance. The current set the *Buzzard* during this interval S. 60° W. 0′·415, the part of which resolved in direction of her course is 0′·086, in opposition to the ship. The *Buzzard's* distance in this interval, given by the patent logs, was 3·02, which multiplied by 1·06 gives for distance through water 3′·2; subtracting 0′·086 from this we have 3′·114 for the distance made good by the *Buzzard* after allowing the corrections *ab*, which agrees very well with the distance checked by the observation from H., and so far their accuracy is confirmed.

Again, taking the point *d* determined by the observation made at 1^h 30^m P.M. from H., and measuring the length of *d*4, we find it to be 3′·83, this is the distance made good by the *Buzzard* from 1^h 30^m P.M. to 2^h 0^m P.M.; her distance through the water during this interval, given by the patent logs, was 3′·55; multiplying by 1·06 we find the corrected distance run through the water $=3'{\cdot}763$; the set of the ship on her course by the current, as before determined, in half-an-hour was 0′·086, applying this to the above we have for the *Buzzard's* distance made good, 3′·85, which also tends to confirm the truth of our assumptions and the accuracy of the corrections.

The sounding between noon and 1^h P.M. was not taken at 0^h 30^m P.M., when she was observed from H., but 10 minutes

later, we therefore place the position of that sounding on the straight line *ab*, by means of its distance from *b*, given by patent log and the set of the ship by current in direction *ab*

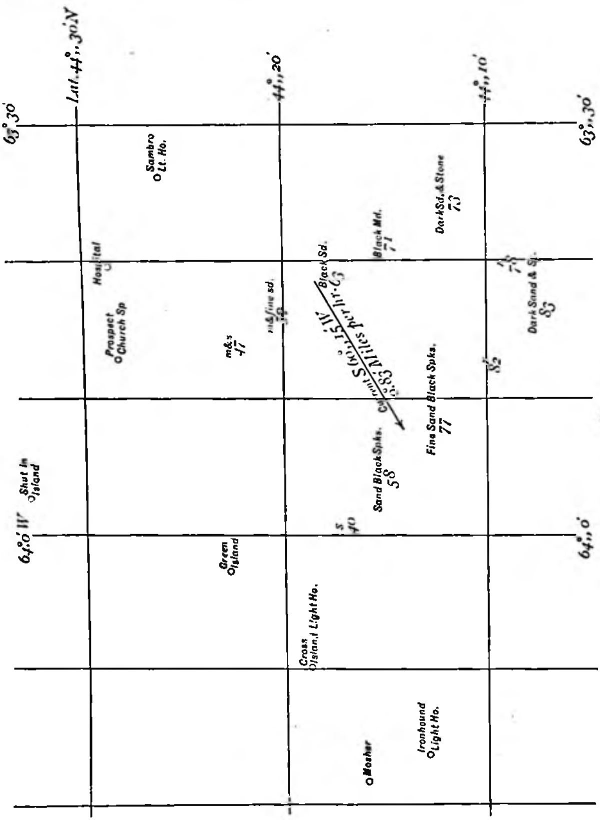


FIG. 74.

in 24 minutes; this distance through the water by patent logs in 24 minutes was 2′·5, multiplying by 1·06 we have distance run through water 2·65, set by current 0′·324: or the distance

of the *Buzzard* from b at $12^h\ 36^m$ was 2'·97; taking this off the scale with a pair of dividers and setting it off from the point b on the straight line ba, we find S the most probable position of the *Buzzard* at $0^h\ 36^m$ P.M. We have some means of roughly testing this by the observations made from H. at $0^h\ 30^m$ P.M., which gave the *Buzzard's* true bearing at that instant S. 0° 38' W. Laying this down on the sheet we have the line H.tT on which the *Buzzard* was at $0^h\ 30^m$ P.M., and t, the point in which it cuts ab, gives the most probable position of the *Buzzard* at that instant. Soon after this the *Buzzard's* engines were eased and stopped to sound, and she had very nearly lost her way. At $0^h\ 36^m$ P.M., when the lead was hove, measuring ts we find it to be 0'·44, which is about the distance the *Buzzard* might be expected to have made in six minutes under the circumstances. The soundings written in their places are shown in Fig. 74, in order that the straight pencil lines, and the points in which they cut, may not be obscured by writing them in Fig. 73.

CHAPTER XIV.

RUNNING SURVEYS.

THIS sort of work requires more tact, experience, and knowledge to do well than ordinary surveying. All the preceding chapters should be mastered before attempting such a duty.

Previous to commencing a survey of this description, and after it is finished, the variations of the ship's compasses must be determined, and during the survey, whenever an observation is made to determine the true bearing of an object on shore from the ship, its magnetic bearing must be observed and the direction of the ship's head noted.

The true bearings of the principal objects on the shore must be obtained by observations made from judiciously selected positions of the ship, whenever practicable.

Sheets of paper for projecting the observations, sketching in the coast line, and for sketching the appearance of the land from selected positions of the ship, must be prepared before beginning the work, and placed on sketching boards ready for immediate use.

A good pocket chronometer is required for deck use; it must be compared with the ship's standard chronometer before commencing work in the morning, and after it is finished in the evening, as well as after the observations on the sun for time have been made during the day.

Two patent logs must be towed from outriggers, one on each quarter of the ship.

The errors of the sounding machines must be ascertained before sounding in the morning, after the work is finished in the evening, and at all times during the day when a good up and down sounding with the line can be obtained, so as to be compared with the depth given by the machine.

The hand lines must be measured when wet and their errors noted in the deck book.

Several surveyors acting in concert are required to make a running survey; the attention of each must be devoted to the special duty he has to perform, and the commander should

keep in mind that, unless a sufficient number of assistants are secured, the ship's time will be wasted, and that each assistant should be appointed to do the work he has the greatest aptitude for.

The surveyor should superintend the whole, determine the starting point, the course to be steered, the rate of the ship (kept uniform), when and what observations are to be made, the intervals between the soundings, and every other essential.

One assistant stationed at a table fixed on deck from which he has a commanding view of the coast, with the sheet for projecting the points properly prepared on a board, pencils and instruments ready for use, must lay off the different positions of the ship on her course, and project the angles observed at each immediately after they have been read off, and sketch in the coast line between the fixed points, noting every peculiarity he sees when they are abeam of the ship. He should have a sextant at hand to take an occasional angle.

Another assistant should sketch the appearance of the land from the different fixed positions of the ship. He should also be prepared to take a sextant angle when necessary.

Another assistant who is a good sextant observer should take with an especially good instrument all the altitudes of the sun, and have another sextant adapted for observing angles between objects on shore, to use when not observing the sun's altitude.

Another assistant should be appointed to attend to the steering of the ship, to take the bearings with the compass, read off the patent logs, and take sextant angles.

The deck chronometer must be taken charge of by another assistant, who will note all the observations in the deck-book, with the corresponding chronometer times *immediately* after they have been taken. This chronometer should be compared with the ship's standard chronometer before beginning work in the morning, immediately after finishing in the evening, and after solar observations for time during the day.

Another assistant should be appointed to look after the soundings, see the lead hove at the proper times, read off the machines, see the soundings properly recorded in the deck-book, and be ready to take a sextant angle.

Each observer should see that his observation is properly recorded in the deck-book immediately after he has read off his instrument.

The ship must be steered on one course as nearly as possible parallel to the mean direction of the coast line. The beginning and end of each course must be fixed in latitude and longitude by solar observations made there, and at other judiciously

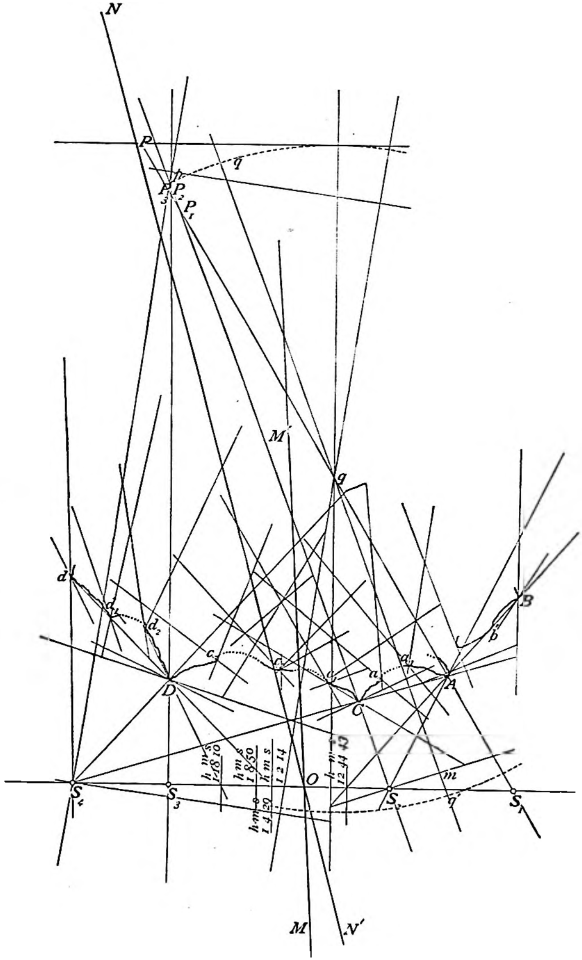


FIG. 75.

selected points on the ship's course, combined with the course and distances given by the patent logs in the manner hereinafter explained. Great care must be taken to keep the ship at a uniform speed, not generally exceeding six miles per hour.

We will take the following example to show more clearly the *modus operandi*. The ship being at anchor in a safe harbour on the coast to be surveyed, at which observations were made for time and latitude, a tide pole erected, and a critical survey of the harbour commenced, during which a station A (see Figs. 75 and 76) was made on the highest part of the point at

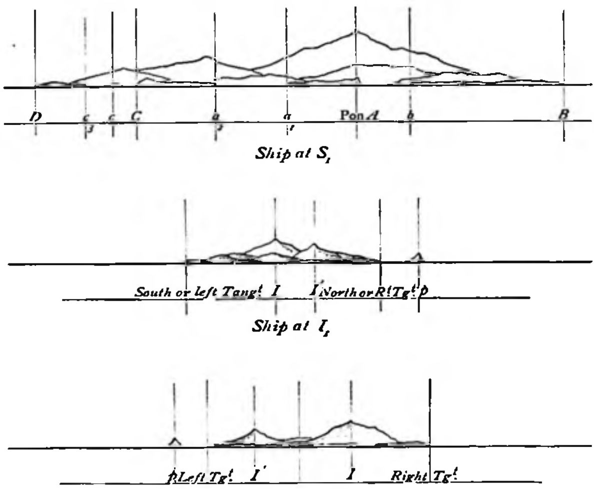


FIG. 76.

the western entrance of the harbour, from which the true bearing of a remarkable high and distant peak, P, was determined by observation to be N. 14° 41′ W., and a remarkable boulder (at the extremity of the point next to the westward of A bore from it S. 88° 2′ W., true), C, was whitewashed so that it might be distinguished from the ship when off shore.

On the 6th August, 1864, the weather being favourable, the ship, after leaving a party encamped on shore to continue the survey of the harbour and observe the tides, weighed early in

the morning and steamed out of harbour to make a running survey of the coast line between the point A and another harbour about sixty miles distant. The mean direction of the coast line between the two harbours was estimated to be about N. 75° 30′ W. true. The variations of the ship's standard compass had just been carefully determined, and from these it was calculated that if the ship was steered W. ¼ N. by her standard compass the true course would be about N. 75° 30′ W. The vessel when sufficiently distant from the shore, with everything in readiness, was placed on the course W. ¼ N. The engines stopped so that she might cross the line P on A slowly, with only sufficient way to keep her head W. ¼ N. All the observations with the times and soundings were noted as soon as they were taken in the ship's deck-book, and are recorded in Table I., where the symbol B denotes the tangent of the eastermost point of land, as seen from the starting point S_1 (see Fig. 75); D the tangent of the cliffs at westernmost extremity of the land. The others will be best understood from the Figures.

TABLE I.

Time by Chronometer.	Course by Standard Compass.	Distance by Patent Logs (Starboard and Port).	Observations.	Soundings. Fathoms.
h. m. s. 11 58 23	W. ¼ N.	S. = 0′·00 P. = 0′·00	Ship at S_1 with P and A in line (see Figs. 75 and 76). Alt. ⊙ 9° 7′ 20″ I.E. + 40″; height of eye above the sea, 18 feet. C - - 30° 9′ A on P D - - 43 25 a_1 - - 9 50 a_2 - - 20 2 A on P - - 29 52 B 8 41 tangt. point on east side of entrance opposite A. Alt. P above A's high { 58′ 0″ on water line by sextant, { 58 40 off. Bearing of P and A in line by standard compass, N. 26° 20′ W. Engines started ahead 25 revolutions and kept uniformly at that rate except when eased and stopped.	22 gr. & sh.
12 8 20	,,	,, ,,	Sounded in - - - - - -	19 sand.
18 20	,,	,, ,,	,, ,, - - - - - -	14 st.
28 20	,,	,, ,,	,, ,, - - - - - -	18 sand.

TABLE I.—*Continued.*

Time by Chronometer.	Course by Standard Compass.	Distance by Patent Logs (Starboard and Port).	Observations.	Soundings. Fathoms.
h. m. s. 31 45	W. ¼ N.	S. = 2′·80 P. = 2′·70	Eased and stopped engines so as to move slowly across the line *P* on *C* with steerage way. Bearing of *P* on *A* by standard compass, N. 16° 40′ W., ship at S_2 (see Fig. 75). Alt. ☉ 15° 46′ 0″ I.E. + 40″ *P* on *C* - 85 59 ... ☉ ” - 44 11 ... *A* ” - 51 8 ... *B* ” - 26 32 ... a_1 ” - 14 12 ... a_2 *D* - 44 35 *P* on *C* c_3 - 33 50 ” c_2 - 23 6 ” c_1 - 10 2 ” Altitude of *C* cliff above its high water line, { 15′ 10″ on / 16 0 off. Went ahead 25 revolutions.	15 rock.
12 38 20	”	” ”	Sounded in - - - - - -	17 s.
44 43	”	” ”	*B* and *b′* in line just shutting in behind *A*.	
48 20	”	” ”	Sounded in - - - - - -	20 gr.
1 2 14	”	” ”	a_2 just shut in behind *C*. c_2 bore by standard compass N. ¼ E. Altitude of cliff at c_3 above its high water line, { 12′ 15″ on / 13 0 off.	
1 4 29	”	” ”	Rocky point *d* just opening out beyond *D*.	
1 8 50	”	” ”	Sand beach at d_1 just opening beyond *D*, and a_1 just shut in behind *C*. Sounded in - - - -	21 gr.
1 18 10	”	” ”	Rocky cliff from *D* to d_2 just opening out beyond D_1. Sounded in - -	22 s. & sh
1 28 20	”	” ”	Sounded in - - - - - -	24 s.
1 32 12	”	S. = 7′·6 P. = 8·0	Stopped engines, going slowly through the water; ship at S_3, - - -	23 s.
			With *P* on *D* bearing, standard compass, N. 2° 20′ E. Obsd. alt. ☉ 28° 15′ 20″ I.E. + 40 *P* on *D* - - 76 31 ... ☉ ” - - 64 13 ... *C* ” - - 66 15 ... *A* ” - - 54 25 ... c_1 ” - - 42 20 ... c_2 ” - - 20 15 ... c_3 d_2 - - 8 50 *P* on *D* d_1 - - 19 2 ” *d* - - 25 21 ”	

TABLE I.—*Continued.*

Time by Chronometer.	Course by Standard Compass.	Distance by Patent Logs (Starboard and Port).	Observations.	Soundings. Fathoms.
h. m. s. 1 38 20	W. ¼ N.	S. = 7'·6 P. = 8·0	Sounded in - - - - - -	25 m.
1 48 25	”	” ”	” - - - - - -	27 s.
1 59 46·6	”	S. = 9'·76 P. = 10·14	Eased and stopped engines; ship at S_1 (Fig. 75). Obsd. alt. $\underline{\odot}$ 33° 34' 30" I.E. + 40" *P* - - 66 40 … |$\odot$ ” - - 63 58 *C* and *A* in line. ” - - 57 22 c_1 ” - - 50 48 c_2 ” - - 38 35 c_3 *D* - - 31 8 *C* *d* - - 73 52 ” d_1 - - 62 49 ” d_2 - - 48 10 ” Bearing *P*, stand. compass, N. 11° E.	28 m.
The observa	tions in	the deck-book	between 2^h and 4^h chron. time are	omitted.
4 0 12·3	W. ¼ N.	S. = 19'·35 P. = 20·05	Engines eased and stopped; ship moving very slowly on her course at S_2. Sounded in - - - Obsd. alt. $\underline{\odot}$ 57° 14' 20" I.E. + 40" *P* 66 40 |$\odot$ no index E.	102 m.
The observa	tions in	the deck-book	are omitted until those made for putt	ing in an
Island,	which ap	peared on the	port bow, were taken.	
5 26 48	W. ¼ N.	S. = 26'·44 P. = 27·28	Engines eased and stopped so as just to keep steerage way; ship at I_1 (Fig. 76). Magnetic bearing of *I*, the highest point of the island, S. 58° 30' W. Obsd. alt. $\underline{\odot}$ 69° 42' 40" I.E. + 40" $\odot$| 91 21 *I*, no I.E. South or left tangent of island a bold rocky point 8° 28' *I*. *I* … 13° 10' *p* the highest point of a detached pinnacle rock. *I* … 3° 50' *I'* top of high hill. ” .. 9 42 north or right tangt. of island.	32 s.
5 57 12	”	” ”	Mag. bearing of I_1 S. 39° 40' W., *I* 9° 55', distant point opening out beyond the north or right tang. before observed.	
6 0 40	”	” ”	Magnetic bearing of *I* S. 36° 20' W., left tangent before observed just shutting in behind a nearer point. 12° 20' *I* and *I'* in line. 23 19 left tangent pinnacle. 25 30 *p*. 27 6 right tangent pinnacle.	

TABLE I.—*Continued.*

Time by Chronometer.	Course by Standard Compass.	Distance by Patent Logs (Starboard and Port).	Observations.	Soundings. Fathoms.
h. m. s. 6 1 12	W. ¼ N.	S. = 29′·2 P. = 30·4	Eased and stopped engines; ship at S_n going very slow.	
			Sounded in - - - - -	22 r.
			Lat. by circum-meridians, 34° 59′ 20″ N.	
6 19 42	„	„ „	Highest point of pinnacle p_1 in line with I, magnetic bearing S. 14° 40′ W., I 11° 16′, distant point just opening to the right of the right tangent of island. Alt. p above its high water line {24′ 45″ on / 25 50 off. Alt. I' above p's high line {33′ 10″ on / 34 20 off} I {37′ 20″ / 38 20}	
6 27 20	„	„ „	Near point in line with and just shutting out the left tangent of the island, 17° 51′ I.	
6 41 3	„	S. = 32′·32 P. = 33·56	Ship at I_2; magnetic bearing of I, S. 13° E., left tangent of island last observed just shutting in behind a nearer point in line with p, 21° 58″ I. ... p 76° 31′ ⊙ no I.E. 29 13 right tangent of 2. 35 23 right tangent of island. I' 12 5 I. Alt. I above its high water line, {32′ 5″ on / 33 20 off. Alt. p above its high water line, {19 50 on / 20 50 off. Alt. I' above high water line of island, {28 0 on / 20 10 off. Tangents of pinnacle rock, {23 8 / 19 37} I	
7 28 2	„	S. = 36′·1 P. = 37·4	Obsd. alt. ⊙ 69° 15′ 40″ I.E. + 40″ Eased and stopped engines; ship at I_3; magnetic bearing I, S. 52° 0′ E. (see Figs. 75 and 76). Alt. ⊙ 61° 16′ 20″ I.E. + 40″ I 87 30 ⊙ no index error. 7 15 right tangt. island. I' 8 28 I Left tangt. island, } 13 10 I p 16 10 „ Tangts. of do., {16 45 / 15 1} „	

TABLE I.—*Continued.*

Time by Chronometer.	Course by Standard Compass.	Distance by Patent Logs (Starboard and Port).	Observations.	Soundings. Fathoms
h. m. s.				
Observations	in deck-	book omitted	until	
8 3 24	W. ¼ N.	S. =38′·8 P. =40·5	Eased and stopped engines; ship at S_6. Sounded in - - - - - Alt. ☉ 56° 58′ 40″ I.E.+40	82 s.
Observations	in deck-	book omitted	until	
11 3 22·8	"	S. =48′·6 P. =50·6	Eased and stopped engines; ship at S_7. Sounded in - - - - - Alt. ☉ 33° 25′ 0″ I.E.+40″	98 m.
Observations	in deck-	book omitted	until	
12 4 49	"	S. =58′·1 P. =60·3	Eased and stopped engines; ship at S_8. Sounded in - - - - - Alt. ☉ 8° 59′ 30″ Index error +40″ Height of eye during all the observations above recorded was 18 feet above the sea.	20 s.
Other obser to the an	vations chorage	in deck-book in the harbo	omitted, after taking which the vessel ur to which she was bound.	proceeded

The sheet for projecting the observations and sketching in the coast line was prepared in the chart room *before* leaving the anchorage as follows. About 1 or 1½ inches from the bottom of the sheet the straight line $S_1S_2S_3S_4$ (see Fig. 75) was drawn right across the sheet to represent the ship's course N. 75° 30′ W. S_1, the position of the ship at starting, at which the peak P of the high distant land is on with the nearer station A; the true bearing of this line, determined from observations made at A, is N. 14° 41′ W., and therefore makes an angle of 60° 49′ with the course $S_1S_2S_4$. At S_1, by means of its chord, make the angle S_4S_1P equal to 60° 49′; consequently, S_1S_4 being the ship's course, S_1 will be the point at which A and P are seen in one.

At a point O in S_1S_4, nearly half way between the margins of the sheet, draw the straight line NON', making the angle $S_4ON = 75° 30′$; then NON' will be the true meridian through O, and at the point O make the angle NOM' equal to 11° 30′ and produce $M'O$ to M; then $M'OM$ will be parallel to the direction of the magnet of the standard compass when the ship is on her course W. ¼ N. The sheet was then placed on a sketching board and was ready for use. The scale of projection should be as large as possible, subject to the condition that the

principal points inshore of the ship *must* come well within the upper margin of the sheet. Fig. 75 is projected on the scale of half an inch to the mile, so that the most distant point, P, may fall well within the margin of a sheet of convenient size, and be sufficiently large to show *how* the projections are made, but in practice a much larger scale should be adopted. The projector from his station on deck lays down from S_1 the angles taken at $11^h\ 58^m\ 3^s$, using a straight edged protractor, referring them to the straight line S_1P already drawn on the sheet, and thus obtains straight lines passing through the points C, D, a_1, a_2, b, and B. At chronometer time, $12^h\ 31^m\ 45^s$, when P came in one with C, the mean of the readings of the patent logs was $2'{\cdot}75$. With a pair of dividers the projector takes $2{\cdot}75$ off the half inch scale and lays it off from S_1 on the ship's course $S_1S_2S_3S_4$, and finds the point S_2, the position of the ship at chronometer time $12^h\ 31^m\ 45^s$. From this position the magnetic bearing of P was N. 16° 40′ W, but from S_1 the observed magnetic bearing of P was N. 26° 20′ W.; therefore the angle at P between S_1 and S_2 is 9° 40′; in S_1P take a point Q about half way between S_1 and P_1 as nearly as the projector can estimate, and from the point Q lay off the angles PQq and S_1Qq' with the straight edged protractor—each equal to 9° 40′ (using the chord of 60°). Draw the straight line qQq'; from q draw qp perpendicular to qQq', using the straight edged protractor; then slip the protractor along the straight line qQq', still keeping its edge perpendicular thereto, until its edge passes through the point S_2 and draw the straight pencil line S_2m along its edge; then S_2m will be perpendicular to qQq'. Make $qp = S_2m$; draw the straight line pP_1CS_2 cutting S_1P in P and S_1C in C, then the angle $S_2P_1S_1$ will $= 9°\ 40'$, and P_1 will be the first approximate position of P, and the point C will be that of the whitewashed boulder C. At the point S_2 from the straight line S_2P_1 thus drawn lay off the angles observed at chronometer time $12^h\ 31^m\ 45^s$, and thus obtain another series of lines passing through the points D, a_1, a_2, A, and B, and cutting the straight lines drawn from S_1 in the first approximate positions of those points, and also straight lines passing through the points c_1, c_2, and c_3. At chronometer time $12^h\ 44^m\ 43^s$ B and a point b' between it and A were in the same straight line and shut in together behind A, so that in this position of the ship A, b', and B were in the same straight line with it. Taking the difference between the two times, we find, supposing the ship's speed to have remained the same since leaving S_2 as whilst moving from S_1 to S_2, that the distance from S_2 to the ship's position at $12^h\ 44^m\ 43^s$ was $0'{\cdot}96$. Laying this distance off from S_2 on the ship's course, we

find her position corresponding to the above time, through which draw a straight line in pencil perpendicular to the course, and along it write the time $12^h\ 44^m\ 43^s$. From this point the projector draws a straight line to B, which should pass through A; he makes a pencil dot half way between A and B as nearly as he can judge by his eye; he also sketches in the coast line lightly between the points he has fixed. Proceeding in the same manner with the observations recorded in the table at the times when they were observed, the other lines defining the points on the shore were projected and the coast line sketched in between the points as they were fixed. The estimated height of the cliffs above the high water line were noted in the sketch, to be afterwards corrected from those also estimated, but for which observations were made from which the heights above the high water could be calculated after the distances from the ship had been determined.

The appearances of these were sketched from the positions S_1, S_2, S_3, S_4, etc., of the ship by the assistant selected for that purpose. Those made from S_1, I_1, and I_3 are given in Fig. 76.

After the day's work was completed, and the ship anchored in the harbour to which she was proceeding, the observations for time and true bearing were calculated, and the course and distances corrected in the following manner.

First we notice that six observations on the sun's altitude for time were made at positions S_1, S_4, S_5, S_6, S_7, and S_8 of the ship recorded in Table I., also circum-meridian altitudes of the sun were taken at S_n, which gave the latitude of that place to be 34° 59′ 20″. We assume that the errors of the patent logs and the current were uniform, whilst the ship was running from S_1 to S_8; and that the ship kept the same course.

Table II. gives the course, and the distances by patent logs measured from S_1, when they read zero, with the differences of latitude, longitude, and time determined therefrom.

TABLE II.

Position of Ship.	From S_1.						Remarks.
	Course.	Distance.	Difference of Latitude.	Dep.	Difference of Longitude.	Difference of Time.	
	° ′	′	′	′	′	s	
S_4	N. 75 30 W.	9·95	2·50	9·63	11·95	47	
S_5	"	19·70	4·93	19·07	23·28	93·1	
S_n	"	29·85	7·46	28·86	35·20	140·8	
S_6	"	39·65	9·92	38·39	45·70	182·8	
S_7	"	49·60	12·45	47·98	57·80	231·2	
S_8	"	59·40	14·90	57·49	70·30	281·2	

To determine the latitudes of S_1, S_4, etc., S_8 from the observed latitude of S_n, we have

Observed latitude of S_n, - -	34° 59′·33 N.
S_1 south of S_n by dead reckoning,	7·46
Latitude of S_1, - - - - -	34° 51′·87 N.
S_4 south of S_n, - - - - -	4′·96
Latitude of S_4, - - - - -	34° 54′·37 N.
S_5 south of S_n, - - - - -	2′·53
Latitude of S_5, - - - - -	34° 56′·80 N.
S_6 north of S_n, - - - - -	2′·46
Latitude of S_6, - - - - -	35° 1′·79 N.
S_7 north of S_n, - - - - -	4′·99
Latitude of S_7, - - - - -	35° 4′·32 N.
S_8 north of S_n, - - - - -	7′·44
Latitude of S_8, - - - - -	35° 6′·77 N.

The results of the calculations from the observed altitude of the sun will alone be given hereafter, but should the reader wish to make the calculations, I have given in Table III. the sun's declination, equation of time, and semi-diameter.

TABLE III.

Chronometer Time.	Sun's Declination.	Equation of Time.	
	North.	+ Ap.T.	
h m s	° ′	m s	
11 58 23	16 33	5 34·3	
1 59 49·7	31·6	33·7	
4 0 12·3	30·3	33·2	Sun's semidiameter, 15′ 48″ 6.
8 3 24	27·7	32·1	
10 3 22·8	26·3	31·5	
12 4 49	25	30·9	

Using the latitudes of S_1, S_4, S_5, etc., S_8 just determined from the observed latitude of S_2, we find from the observations

Sun's hour angle at	S_1,	- - -	6^h	0^m	42^s
" "	S_4,	- - -	4	0	3
" "	S_5,	- - -	2	0	21
" "	S_6,	- - -	2	1	24
" "	S_7,	- - -	4	0	37·6
" "	S_8,	- - -	6	1	18

Let λ be the error in the observed latitude of S_n expressed in miles, + when measured towards the north or elevated pole; λ' the hourly error in the difference of latitude deduced from the dead reckoning and expressed in the same manner; e the hourly error in the difference of longitude derived from the dead reckoning expressed in seconds of time, positive towards the west.

When calculating the hour angles of the sun it was found that when the sun's hour angle was 6^h, an increase of one mile in the latitude of the ship gave an increase of $1^s{\cdot}2$ to the hour angle; that when the sun's hour angle was 4^h the same increase in the latitude diminished the sun's hour angle $0^s{\cdot}24$, and that when the sun's hour angle was 2^h the same increase in the latitude of the ship diminishes the hour angle $2^s{\cdot}40$.

Hence the error in the latitude of

$$\begin{aligned}
S_1 &= \lambda - 6\lambda'.\\
S_4 &= \lambda - 4\lambda'.\\
S_5 &= \lambda - 2\lambda'.\\
S_6 &= \lambda + 2\lambda'.\\
S_7 &= \lambda + 4\lambda'.\\
S_8 &= \lambda + 6\lambda'.
\end{aligned}$$

Therefore the sun's hour angle at

$$\begin{aligned}
S_1 &= 6^h\ 0^m\ 42^s\ + 1^s{\cdot}2\ (\lambda - 6\lambda').\\
S_4 &= 4\ \ 0\ \ 3\ - 0{\cdot}24(\lambda - 4\lambda').\\
S_5 &= 2\ \ 0\ \ 21\ - 2{\cdot}46(\lambda - 2\lambda').\\
S_6 &= 2\ \ 1\ \ 24{\cdot}6 - 2{\cdot}46(\lambda + 2\lambda').\\
S_7 &= 4\ \ 0\ \ 37{\cdot}6 - 0{\cdot}24(\lambda + 4\lambda').\\
S_8 &= 6\ \ 1\ \ 18\ + 1{\cdot}2\ (\lambda + 6\lambda').
\end{aligned}$$

The difference of longitude between

$$\begin{aligned}
S_1 \text{ and } S_4 &= 47^s + 2e.\\
S_1 \text{ and } S_5 &= 1^m\ 33{\cdot}1 + 4e.\\
S_1 \text{ and } S_6 &= 3\ \ 2{\cdot}8 + 8e.\\
S_1 \text{ and } S_7 &= 3\ \ 51{\cdot}7 + 10e.\\
S_1 \text{ and } S_8 &= 4\ \ 41{\cdot}2 + 12e.
\end{aligned}$$

Hence

$$\begin{array}{llll} \text{Appt. time at } S_1, \text{ 5th Aug.} & = 24^h - (6^h\ 0^m\ 42^s) - 1^s{\cdot}2(\lambda - 6\lambda') \\ \text{Eqn. of time} \quad ,, & ,, \quad + 5^m\ 34^s{\cdot}3. \\ \text{Mean time} \quad ,, & ,, \quad = 18^h\ 4^m\ 52^s{\cdot}3 - 1^s{\cdot}2(\lambda - 6\lambda'). \end{array}$$

In like manner

$$\begin{array}{lll} \text{Mean time of obs. at } S_4, \text{ 5th Aug.} & = 20^h\ 5^m\ 30^s{\cdot}7 + 0^s{\cdot}24(\lambda - 4\lambda'). \\ \quad ,, \qquad ,, \qquad S_5, \text{ 5th Aug.} & = 22\ \ 5\ \ 12\ {\cdot}2 + 2\ {\cdot}46(\lambda - 2\lambda') \\ \quad ,, \qquad ,, \qquad S_6, \text{ 6th Aug.} & = \ 2\ \ 6\ \ 56\ {\cdot}7 - 2\ {\cdot}46(\lambda + 2\lambda'). \\ \quad ,, \qquad ,, \qquad S_7, \text{ 6th Aug.} & = \ 4\ \ 6\ \ \ 9\ {\cdot}1 - 0\ {\cdot}24(\lambda + 4\lambda'). \\ \quad ,, \qquad ,, \qquad S_8, \text{ 6th Aug.} & = \ 6\ \ 6\ \ 48\ {\cdot}9 + 1\ {\cdot}2\ (\lambda + 6\lambda'). \end{array}$$

Reducing these times to the meridian of S_1, we have

Mean time at S_1 for observation made then at chronometer time $11^h\ 58^m\ 23^s$

$$= 18^h\ 4^m\ 52^s{\cdot}3 - 1^s{\cdot}2(\lambda - 6\lambda').$$

Mean time at S_1 for observation made at S_4 at chronometer time $1^h\ 59^m\ 46^s{\cdot}6$

$$\begin{aligned} &= 20^h\ 5^m\ 30^s{\cdot}7 + 0^s{\cdot}24(\lambda - 4\lambda') + 47^s + 2e \\ &= 20\ \ 6\ \ 17\ {\cdot}7 + 0\ {\cdot}24(\lambda - 4\lambda') + 2e. \end{aligned}$$

Mean time at S_1 for observation made at S_5 at chronometer time $4^h\ 0^m\ 12^s{\cdot}3$

$$\begin{aligned} &= 22^h\ \ 5^m\ 12^s{\cdot}2 + 2^s{\cdot}46(\lambda - 2\lambda') + 1^m\ 33^s + 4e \\ &= 22\ \ \ 6\ \ 45\ {\cdot}2 + 2\ {\cdot}46(\lambda - 2\lambda') + 4e. \end{aligned}$$

Mean time at S_1 for observation made at S_6 at chronometer time $8^h\ 3^m\ 24^s$

$$\begin{aligned} &= \ 2^h\ \ 6^m\ 57^s{\cdot}6 - 2^s{\cdot}46(\lambda + 2\lambda') + 3^m\ 2^s{\cdot}8 + 8e \\ &= \ 2\ \ 10\ \ \ 0\ {\cdot}4 - 2\ {\cdot}46(\lambda + 2\lambda') + 8e. \end{aligned}$$

Mean time at S_1 for observation made at S_7 at chronometer time $10^h\ 3^m\ 22{\cdot}8^s$

$$\begin{aligned} &= \ 4^h\ \ 6^m\ \ 9^s{\cdot}1 - 0{\cdot}24^s(\lambda + 4\lambda') + 3^m\ 51^s{\cdot}2 + 10e \\ &= \ 4\ \ 10\ \ \ 0\ {\cdot}3 - 0{\cdot}24\ (\lambda + 4\lambda') + 10e. \end{aligned}$$

Mean time at S_1 for observation made at S_8 at chronometer time $12^h\ 4^m\ 49^s$

$$\begin{aligned} &= \ 6^h\ \ 6^m\ 48^s{\cdot}9 + 1^s{\cdot}2\ (\lambda + 6\lambda') + 4^m\ 41^s{\cdot}2 + 12e \\ &= \ 6\ \ 11\ \ 30\ {\cdot}1 + 1\ {\cdot}2\ (\lambda + 6\lambda') + 12e. \end{aligned}$$

Let E be the error of the chronometer fast of mean time at S_1 at chronometer time $11^h\ 58^m\ 23^s$, which was gaining daily $2^s{\cdot}4$ on mean time.

Therefore its error on S_1 mean time

$$
\begin{array}{lllll}
\text{at chronometer time} & 1^h & 59^m & 46^s{\cdot}6 & = E + 0^s{\cdot}2 \\
\quad \text{,,} \qquad \text{,,} & 4 & 0 & 12\cdot3 & = E + 0\cdot4 \\
\quad \text{,,} \qquad \text{,,} & 8 & 3 & 24 & = E + 0\cdot8 \\
\quad \text{,,} \qquad \text{,,} & 10 & 3 & 22\cdot8 & = E + 1\cdot0 \\
\quad \text{,,} \qquad \text{,,} & 12 & 4 & 49 & = E + 1\cdot2
\end{array}
$$

Therefore $23^h\ 58^m\ 23^s - E =$ mean time at S_1 when observation there was taken

$$= 18^h\ 4^m\ 52^s{\cdot}3 - 1^s{\cdot}2(\lambda - 6\lambda')$$

$$\therefore \quad E - 5^h\ 53^m\ 30^s{\cdot}7 = 1^s{\cdot}2(\lambda - 6\lambda') \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In a similar manner equating the two expressions for S_1 mean time at which the observations at S_4, S_5, S_6, S_7, and S_8 were taken,

$$
\begin{array}{ll}
\text{those at } S_4 \text{ give} & 5^h\ 53^m\ 28^s{\cdot}7 - E = 0^s{\cdot}24(\lambda - 4\lambda') + \ \ 2e \ldots (2) \\
\quad \text{,,} \quad S_5 \quad \text{,,} & 5\ \ 53\ \ 26\cdot7 - E = 2\cdot46(\lambda - 2\lambda') + \ \ 4e \ldots (3) \\
\quad \text{,,} \quad S_6 \quad \text{,,} & E - 5\ \ 53\ \ 22\cdot8 \quad = 2\cdot46(\lambda + 2\lambda') - \ \ 8e \ldots (4) \\
\quad \text{,,} \quad S_7 \quad \text{,,} & E - 5\ \ 53\ \ 21\cdot5 \quad = 0\cdot24(\lambda + 4\lambda') - 10e \ldots (5) \\
\quad \text{,,} \quad S_8 \quad \text{,,} & 5\ \ 53\ \ 17\cdot7 - E = 1\cdot24(\lambda + 4\lambda') + 12e \ldots (6)
\end{array}
$$

Here we have six equations involving the four unknown quantities E, λ, λ', and e, and to obtain their most probable values we proceed as follows.

$$
\begin{array}{lrl}
\text{Add (1) and (6)} \quad - \quad - & -13^s = & 2^s{\cdot}4 \times \lambda + 12e \ldots (7) \\
\text{,, (2) ,, (5)} \quad - \quad - \quad - & 7\cdot2 = & 0\cdot48 \times \lambda - \ \ 8e \ldots (8) \\
\text{,, (3) ,, (4)} \quad - \quad - \quad - & 3\cdot9 = & 4\cdot92 \times \lambda - \ \ 4e \ldots (9) \\
\text{subtract (6) from (1)} & 2E - 11^h\ 46^m\ 48^s{\cdot}4 = & -14\cdot4 \ \times \lambda' - 12e \ (10) \\
\quad \text{,, (2) ,, (5)} & 2E - 11\ \ 46\ \ 50\cdot2 = & 1\cdot92 \times \lambda' - 12e \ (11) \\
\quad \text{,, (3) ,, (4)} & 2E - 11\ \ 46\ \ 49\cdot5 = & 9\cdot84 \times \lambda' - 12e \ (12) \\
\quad \text{,, (11) ,, (10)} \quad - \quad - & -1\cdot8 = & 16\cdot32 \times \lambda' \ \ldots\ldots\ldots (a) \\
\quad \text{,, (10) ,, (12)} \quad - \quad - & -1\cdot1 = & 24\cdot24 \times \lambda' \ldots\ldots\ldots\ldots (b) \\
\quad \text{,, (11) ,, (12)} \quad - \quad - & 0\cdot7 = & 7\cdot92 \times \lambda' \ldots\ldots\ldots\ldots (c)
\end{array}
$$

Taking these different values of λ' to be good in proportion to the respective coefficients of λ', multiply (a) by 2, (b) by 3, and (c) by 1, and add; the resulting equation will give the most probable value of λ' we can obtain from the observations.

$$
\begin{array}{lrl}
(a) \text{ multiplied by 2 gives} & 32{\cdot}64 \times \lambda' = & -3{\cdot}6 \\
(b) \quad \text{,,} \qquad 3 \quad \text{,,} & 72{\cdot}72 \times \lambda' = & -3{\cdot}3 \\
(c) \quad \text{,,} \qquad 1 \quad \text{,,} & 7{\cdot}92 \times \lambda' = & 0{\cdot}7 \\
\hline
 & 113{\cdot}3 \ \times \lambda' = & -6{\cdot}2
\end{array}
$$

$$\therefore \qquad \lambda' = -\frac{62}{1133} = -0'{\cdot}055$$

Dividing (7) by 12 we have $e+0^s{\cdot}2 \times \lambda = -1^s{\cdot}08$
„ (8) „ 8 „ $e-0{\cdot}06 \times \lambda = -0{\cdot}9$
„ (9) „ 4 „ $e-1{\cdot}23 \times \lambda = -0{\cdot}98$

Eliminating e between each pair of these equations

$$\begin{aligned} 0{\cdot}26 \times \lambda &= -0{\cdot}18 \\ 1{\cdot}43 \times \lambda &= -0{\cdot}10 \\ 1{\cdot}17 \times \lambda &= \quad 0{\cdot}08 \end{aligned}$$

Multiply the first by 1, the second by 6, and the last by 4,

$$\begin{aligned} 0{\cdot}26 \times \lambda &= -0{\cdot}18 \\ 8{\cdot}58 \times \lambda &= -0{\cdot}60 \\ 4{\cdot}68 \times \lambda &= \quad 0{\cdot}32 \end{aligned}$$

Adding, - - $13{\cdot}52 \times \lambda = -0{\cdot}46$

$$\therefore \qquad \lambda = -\frac{0{\cdot}46}{13{\cdot}52} = -0'{\cdot}034$$

Substituting this value of λ in (7), (8), and (9) we have

$$\begin{aligned} 12e &= -13^s \ +0^s{\cdot}09 = -12{\cdot}91 \\ 8e &= -\ \ 7{\cdot}2 - 0{\cdot}02 = -\ \ 7{\cdot}22 \\ 4e &= -\ \ 3{\cdot}9 - 0{\cdot}17 = -\ \ 4{\cdot}07 \end{aligned}$$

Multiplying the first by 3, the second by 2, and the last by 1, we have

$$\begin{aligned} 36e &= -38^s{\cdot}73 \\ 16e &= -14{\cdot}44 \\ 4e &= -\ \ 4{\cdot}07 \\ \hline 56e &= -57{\cdot}24 \end{aligned}$$

$$e = -\frac{57{\cdot}24}{56} = -1^s{\cdot}022$$

Adding equations (10), (11), and (12),

$$6E - 35^h\,20^m\,28^s{\cdot}1 = -2^s{\cdot}64 \times \lambda' - 36e$$

$$\begin{aligned} \therefore \qquad E &= 5\ \ 53\ \ 24{\cdot}7 - 0^s{\cdot}44\lambda' - 6e \\ &= 5\ \ 53\ \ 24{\cdot}7 + 0{\cdot}02 + 6^s{\cdot}13 \\ &= 5\ \ 53\ \ 30{\cdot}85 \end{aligned}$$

Also

$$\begin{aligned} \lambda - 6\lambda' &= -0'{\cdot}034 + 6 \times 0'{\cdot}055 = \quad 0'{\cdot}3 \\ \lambda + 6\lambda' &= -0{\cdot}034 - 0'{\cdot}33 \qquad = -0{\cdot}36 \\ \lambda - 4\lambda' &= -0{\cdot}034 + 4 \times 0'{\cdot}055 = \quad 0{\cdot}19 \\ \lambda + 6\lambda' &= -0{\cdot}034 - 0'{\cdot}22 \qquad = -0{\cdot}25 \\ \lambda - 2\lambda' &= -0{\cdot}034 + 2 \times 0'{\cdot}055 = \quad 0{\cdot}08 \\ \lambda + 2\lambda' &= -0{\cdot}034 - 0'{\cdot}11 \qquad = -0{\cdot}14 \end{aligned}$$

Hence latitude $S_1 = 34° \ 51'{\cdot}87$ N. $+ 0'{\cdot}3$
$= 34 \ \ 52{\cdot}17$ N.
" $S_8 = 35 \ \ 6{\cdot}77 - 0'{\cdot}36$ N.
$= 35 \ \ 6{\cdot}41$ N.
" $S_4 = 34 \ \ 54{\cdot}37 + 0'{\cdot}19$ N.
$= 34 \ \ 54{\cdot}56$ N.
" $S_7 = 35 \ \ 4{\cdot}32 - 0'{\cdot}25$ N.
$= 35 \ \ 4{\cdot}07$ N.
" $S_5 = 34 \ \ 56{\cdot}8 \ + 0'{\cdot}08$ N.
$= 34 \ \ 56{\cdot}88$ N.
" $S_6 = 35 \ \ 1{\cdot}79$ N. $- 0'{\cdot}14$
$= 35 \ \ 1{\cdot}65$ N.

The effect of the current, errors of patent logs, and bad steerage between S_1 and S_8 is $12e$ in longitude and $12\lambda'$ in latitude.

$e = -1^s{\cdot}022$, $\therefore$ $12\,e = -12^s{\cdot}264 = 3'{\cdot}066$ to eastward in long.
$\lambda' = -0\,{\cdot}055$, $\therefore$ $12\lambda' = -\ 0'{\cdot}66$ to south.

log diff. long. $3'{\cdot}066$ - - - -	0·4866	log dep.	0·4001
log cos 34° 59′ - - - - - - -	9·9135	log cosec co.	0·0199
log dep. 2·512 to eastward, - -	0·4001	log dist. (2·63)	0·4200
" diff. lat., 0·66 -	1·8195		
log tan. course, S. 72° 46·5 E., -	10·5806		

Therefore the general effect of all the errors in the twelve hours was S. 72° 46′·5 E. 2′·63 miles.

	N.	S.	E.	W.
Course and distance from S_1 to S_8 by D.R. was N. 75° 30′ W. 59′·4 miles, -	14·9			57′·5
By current and errors, - - -		0·7	2·5	
	14·2			55·0

log. 14·2, - -	1·1523	log cosec co.	0·0140
log. 55·0, - -	1·7404	- -	1·7404
log tan co. 75°31′30″	10·5881		1·7544 log dist. 56′·8 miles

Therefore the patent log distances must be multiplied by $\frac{568}{594}$ or by 0·956 to give the distances actually made good by the ship between S_1 and S_8.

To determine PS_1 from the observations made at S_2, S_3, S_4, and S_5.

At S_2 mean reading of patent logs gives $S_1S_2 = 2'{\cdot}75$; multiplying this by 0·956 we find the corrected distance $S_1S_2 = 2'{\cdot}63$. Observed alt., ☉ 15° 46′ 40″ gives

True bearing of sun's centre from S_2,	N.	80° 41′ E.
Horizontal angle, „ „ P,		86 6 ☉
Therefore true bearing of P from S_2,	N.	5 25 W.
Direction, S_1S_2, - - - -	N.	75 31·5 W.
True bearing P from S_1, - -	N.	14 41 W.

∴ angle S_2PS_1	= 9° 16′	log 2·63 - - -	0·41996
„ PS_1S_2	= 60 50·5	log cosec 9° 16 -	0·79309
„ S_1S_2P	= 109 53·5	log sin 109 53·5 -	9·97774
		log $S_1P(15'{\cdot}37)$ -	1·18679

At S_3 mean reading of patent logs gives $S_1S_3 = 7'{\cdot}8$.

∴ corrected length $S_1S_3 = 7'{\cdot}46$. Sun's true alt. 28° 26′ gives

True bearing of sun's centre,	N.	89° 18′ E.
Horizontal angle „ „ P,		75 29 ☉
True bearing of P, - -	N.	13 49 E.
True bearing of P from S_3, -	N.	13 49 E.
„ „ S_1, -	N.	14 41 W.

∴ angle S_3PS_1 -	= 28° 30′	log 7′·46 - - -	0·8727
„ PS_1S_3 -	= 60 50·5	log sin 90° 39′·5 -	9·9999
„ S_1S_3P -	= 90 39·5	log cosec 28° 30′ -	0·3213
		log $S_1P(15'{\cdot}63)$ - - - - -	1·1939

At S_4 mean reading patent logs gives $S_1S_4 = 9'{\cdot}95$, which, corrected, makes $S_1S_4 = 9'{\cdot}51$.

At S_4 true bearing of P was	N.	24° 6′·2 E.
„ S_1 „ „	N.	14 41 W.

∴ angle S_4PS_1 -	= 38° 47′·2	log 9′·51 - - -	0·9782
„ PS_1S_4 -	= 60 50·5	log sin 80° 22′·3 -	9·9938
„ S_1S_4P -	= 80 22·3	log cosec 38° 47′·2 -	0·2031
		log $S_1P(14'{\cdot}97)$ - - - - -	1·1751

At S_5 mean of patent log readings gives $S_1S_5 = 19'{\cdot}7$. Therefore corrected distance $S_1S_5 = 18'{\cdot}83$.

At S_5 true bearing of P was N. 54° 53′ E.
" S_1 " " N. 14 41 W.

∴ angle S_5PS_1 -	$= 69°\ 34'$	log 18′·83 - -	1·2748
" PS_1S_5 -	$= 60\ \ 50{\cdot}5$	log sin 49° 35′·5 -	9·8816
" PS_5S_1 -	$= 49\ \ 35{\cdot}5$	log cosec 69° 34′ -	0·0282
	log S_1P (15′·3) -	- - - -	1·1846

We have therefore four values of S_1P determined from observations made at four different positions of the ship, and these should be meaned in proportion of the relative values of the different determinations. The principal error in the true bearings calculated from the observed altitudes and corrected latitudes will result from an error in the observed altitude of the sun. We find that

at S_2 an error of 1′ in the obsvd. alt. gives 0′·4 error in the bearing.
" S_3 " " " " " 0·7 " "
" S_4 " " " " " 0·72 " "
" S_5 " " " " " 1·6 " "

Suppose the error in reading off the patent logs to have for its average size δ miles, regardless of sign, we must estimate as nearly as possible the errors on S_1P arising from the errors in the observed altitudes of the sun in terms of δ.

In the latitude of the ship when the sun's declination is $16\frac{1}{2}°$ N. and the sun's altitude 39°, an error of half a minute in the sun's altitude gives an error of half a minute in the sun's true bearing calculated from it, and an object one mile distant from the observer will in consequence be displaced 0′·0003 from its correct position in a direction perpendicular to the straight line joining the observer and the object; half a minute may be taken as the average size of an error likely to be made in observing the sun's altitude; and, taking the average size of an error likely to be made in reading off the patent logs to be 0′·0072, this is 24 times as large as the former, and therefore if δ represent the average size of the errors made in reading off the patent logs, $\frac{\delta}{24}$ will represent the displacement of an object one mile distant from the observer, in a direction perpendicular to the straight line joining the object and the observer, due to an error of half a minute in the true bearing, and is therefore the size of the error likely to be made under such circumstances when the sun's altitude is 39°.

Hence for sun's altitude at S_2 observation size of error on objects at one mile distant from observer, $\Big\} = \frac{\delta}{60}$

For sun's altitude at S_3 observation, - - - - - $= \frac{\delta}{34}$

" " S_4 " - - - - - $= \frac{\delta}{30}$

" " S_5 " - - - - - $= \frac{\delta}{15}$

Therefore average size of an error in S_1P arising from the errors in the observation made at

$$S_2 = \frac{\delta \sin S_2}{\sin S_2PS_1} + \frac{\delta}{60}S_1S_2\frac{\sin S_1}{\sin^2 S_2PS_1}.$$

Since δ will enter as a multiplier of each error, the *proportional* sizes of which it is our object to compare, dividing each by δ we shall have proportionally

$$\text{Size of } S_2 \text{ error} = \frac{\sin(109^\circ\ 53'\cdot 5)}{\sin(9^\circ\ 16')} + \frac{2\cdot 63}{60}\frac{\sin(60^\circ\ 50'\cdot 5)}{\sin^2(9^\circ\ 16')}$$
$$= 5\cdot 84 + 0\cdot 15 = 5\cdot 99.$$

$$\begin{aligned} \text{Proportional size of } S_3 \text{ error} &= 2\cdot 1 + 0\cdot 8 = 2\cdot 9 \\ S_4 \quad\quad &= 1\cdot 6 + 0\cdot 7 = 2\cdot 3 \\ S_5 \quad\quad &= 0\cdot 8 + 1\cdot 2 = 2\cdot 0 \end{aligned}$$

Taking the values of each determination to be inversely proportional to the sizes of the errors they are liable to, they will be respectively proportional to $\frac{1}{60}$, $\frac{1}{29}$, $\frac{1}{23}$, and $\frac{1}{20}$, or as 33 : 69 : 88 : 100.

$$\begin{aligned} \text{Now } S_2 \text{ observations give } S_1P &= 15'\cdot 37 = 15'\cdot 3 + \cdot 07 \\ S_3 \quad\text{"} \quad\quad\quad\text{"} \quad &= 15\cdot 63 = 15\cdot 3 + \cdot 33 \\ S_4 \quad\text{"} \quad\quad\quad\text{"} \quad &= 14\cdot 97 = 15\cdot 3 - \cdot 33 \\ S_5 \quad\text{"} \quad\quad\quad\text{"} \quad &= 15\cdot 30 = 15\cdot 3 \end{aligned}$$

To mean these proportionally to the above values,

$$\begin{array}{rrr} \cdot 0 \times 100 = & & 0'\cdot 0 \\ +\cdot 07 \times 33 = & + & 2\cdot 31 \\ +\cdot 33 \times 69 = & + & 22\cdot 77 \\ -\cdot 33 \times 88 = & - & 29\cdot 04 \\ \hline 290 & - & 3\cdot 96 \\ \hline & - & 0\cdot 014 \end{array}$$

Mean value $S_1P = 15'\cdot 3 - 0'\cdot 014 = 15'\cdot 286$.

We can now make the final projection and sketch the coast line in between the points from the rough sheets.

We therefore take a sheet of sufficient size to embrace the

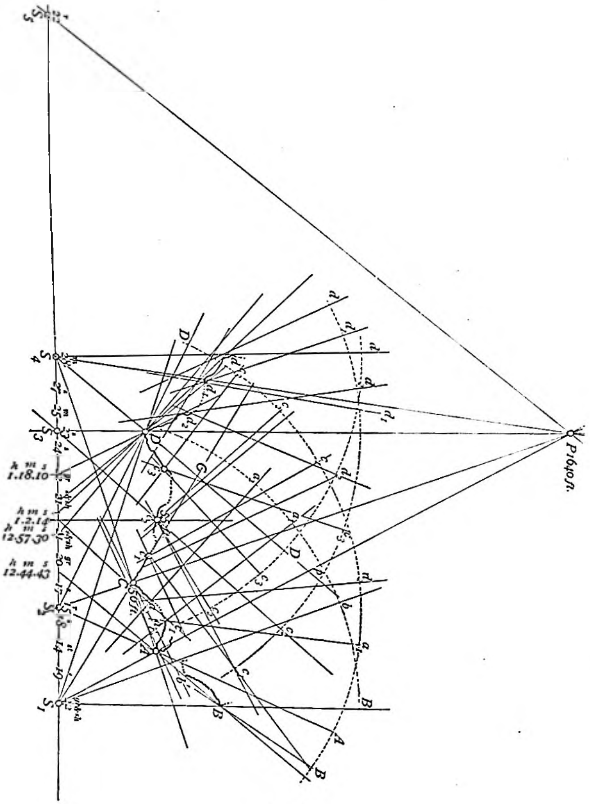


Fig. 77.

most distant inshore points and lay it flat on the chart room table, and with a long steel straight edge rule a straight pencil line right across the sheet parallel or nearly so to its lower

edge, and about 1½ inches distant from it, to represent the ship's course. See Fig. 77, where S_1, S_2, ... S_5 is the ship's course; S_1 is taken a sufficient distance from the margin of the paper to enable the straight line S_1B to lie well within the margin of the paper throughout its length. The scale of projection in the figure is only half an inch to the mile in order to keep the figure at a convenient size for publication, but the projection should be made on as large a scale as possible and certainly not less than two inches to the mile. At S_1 make the angle $S_5S_1P=60°\ 56'\cdot5$ (see Chap. IX.), the chord being calculated for a radius equal to a multiple of 5 inches, which is nearest to the length of S_1P on the scale of construction; make $S_1S_2=2'\cdot63$ (miles), $S_1S_3=7'\cdot46$, $S_1S_4=9'\cdot51$, $S_1S_5=18'\cdot83$, and $S_1P=15'\cdot29$. All the calculated distances must be laid down immediately one after another with beam compasses from the same scale, so as to guard against any change in the condition of the paper or of the scale between the first and last laid off distance; after which all the projections must be made by laying off the angles from the points representing the positions of the ship at the times the angles were respectively taken, in the following manner. From S_1P as a centre with radius S_1p, which in the figure is the chord of 60° taken off a 6 inch scale, describe the dotted circular arc DCa_2a_1bB. Take from the same scale the chord of 43° 25′ with a pair of compasses, placing the point of one leg at p, the point in which the circular arc cuts S_1P, describe with the point of the other leg a small circular arc cutting the former arc in the point D, join S_1D with a straight line, which produced, if necessary, will pass through the object denoted by D in the deck angle book. In the same way, taking off the chords of the other angles in succession, we obtain straight lines passing through the points denoted by C, a_2, etc., and B; in a similar manner, by laying off from S_2, S_3, S_4, and S_5, the angles observed from the ship at the positions they respectively represent, we obtain other straight lines passing through the objects on shore and the points where two straight lines passing through the same objects cut is the position of the object on the paper. When more than two angles have been observed from different positions of the ship, the corresponding straight lines passing through the same object ought to cut each other in one point. This does not generally happen, but the distances between the respective positions of the same point ought to be small, and the projector assumes a point within the figure that would be formed by joining the different points by straight lines, and having reference to the respective values of each cut, to represent the correct position of the object. Referring to Table we see that at chronometer time $12^h\ 44^m\ 43^s$, B and b'

were in one, just shutting in behind A. To find the position of the ship at this time, at $12^h\ 31^m\ 45^s$ the ship was at S_2, $2'{\cdot}63$ from S_1, and at $1^h\ 33^m\ 12^s$ the ship was at S_3, $7'{\cdot}46$ from S_1; hence the distance $S_2S_3 = 4'{\cdot}83$ was run in $61^m\ 27^s$. Now, at $12^h\ 44^m\ 43^s$ the ship had passed S_2, $12^m\ 58$, and therefore the distance from S_2 to the ship at that time $= \dfrac{12^m\ 58^s}{61^m\ 27^s} \times 4'{\cdot}83$ $= 1'{\cdot}02$; take this length from the half-inch scale and lay it off on S_2S_3 from S_2 towards S_3, and through the point thus found draw a straight pencil line perpendicular to S_2S_3, and along it write $12^h\ 44^m\ 43^s$, draw a straight line from this point to A and produce it; b' and B should be on this line, b' being half way between A and B. In a similar manner the ship's positions, corresponding to the times by chronometer, $12^h\ 57^m\ 30^s$, $1^h\ 2^m\ 14^s$, and $1^h\ 18^m\ 10^s$, were determined as well as the positions of the soundings. The coast line was then sketched in between the fixed points from the rough sheet.

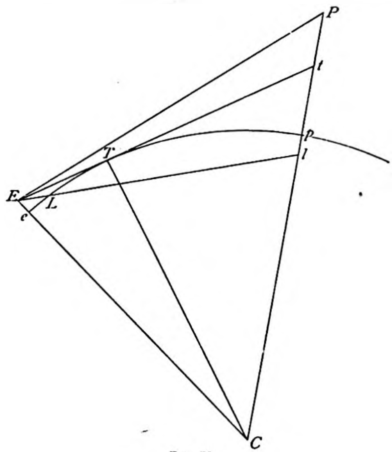


Fig. 78.

To determine the height of the objects above the sea level from the observed angles of elevation.

Let eTp (Fig. 78) be the curve made by a plane passing through E, the eye of an observer, C the centre of the earth, P the top of a distant peak, and cutting the surface of the earth to what we may suppose to be the arc of a great circle, of which C is the centre; join EC, cutting the earth's surface in e, and PC, cutting it in p, from E draw the straight line ETt, touching the earth's surface in T and cutting Pp in t; let L be any point in the arc eLT, join EL and produce it to cut PpC in l.

Let $Ee = h$ feet, $ETt = d$ miles, $ET = x$ miles, $EL = y$ miles, and $2eC = D$ miles.

$$ET^2 = Ee \times eC + EC);$$

$$\therefore \qquad x^2 = \frac{hD}{6000} \text{ very nearly,}$$

$$D \sin 1' = 2 \text{ miles};$$

$$\therefore \qquad x^2 = \frac{h}{3000 \sin 1'} \qquad \text{.........................(1)}$$

Let m be the number of minutes in the angle TEP, considered so small that we may put $Pt = 6000\ dm \sin 1''$ without any practical error, when Pt is expressed in feet;

similarly $$Tt^2 = \frac{tp}{3000 \sin 1'};$$

$$\therefore \qquad tp = Tt^2 \times 3000 \sin 1',$$

where tp is expressed in feet and Tt in miles; but $tp = d - x$, $\therefore tp = (d-x)^2\ 3000 \sin 1'$,

hence $Pp = Pt + tp$

$$= 6000\ dm \sin 1' + (d-x)^2\ 3000 \sin 1'$$

$$= (m + \tfrac{1}{2}d)\, d\, 6000 \sin 1' - 2\, d\sqrt{h \times 3000 \sin 1'} + h \ldots (2)$$

In triangle ELT, angle $TEL = ETL\ \dfrac{LT}{EL}$ very approximately, since the angles are very small $= (x-y)\ \dfrac{x-y}{y}$ very approximately. If L be a point in the high water line of the land, the altitude above which was taken by the sextant, then m in expression (2) must be replaced by $m - \dfrac{(x-y)^2}{y}$, and we shall have

$$Pp = \left(m - \frac{(x-y)^2}{y} + \frac{d}{2}\right) d6000 \sin 1' - 2d\sqrt{h \times 3000 \sin 1'} + h$$

$$= \left(m + \frac{d}{2} - y + 2x\right) d \times 6000 \sin 1' - 2d\sqrt{h \times 3000 \sin 1'} - h\left(\frac{2d}{y} - 1\right)$$

$$= \left(m + \frac{d}{2} + \overline{x-y}\right) d \times 6000 \sin 1' - h\left(\frac{2d}{y} - 1\right) \text{.........................(3)}$$

where h is the height of the observer's eye above the sea, m the number of minutes in the angle of elevation of P above the high water line of land y miles distant, and d the number of miles P is from the observer, x the distance of the sea horizon is from him. I need hardly remark that y can never exceed x. In the foregoing example $h = 18$ feet,.

$$\therefore \qquad x=\sqrt{\frac{h}{3000 \sin 1'}}=\sqrt{\frac{18}{3000 \sin 1'}}=\sqrt{\frac{6}{1000 \times \sin 1'}}$$

log sin 1′	-	$=6{\cdot}463726$
log sin 1000	-	$=3{\cdot}000000$
		$9{\cdot}463726$
log 6 - -	-	$0{\cdot}778151$
		$2)1{\cdot}314425$
log $x(=4{\cdot}542)$	-	$0{\cdot}657212$, $\therefore x=4{\cdot}542$ miles.

TABLE IV.

Dist. Miles.	$d \times 6000 \sin 1'$.	Dist. Miles.	$d \times 6000 \sin 1'$.	Height of Eye. Feet.	x Miles.	Height of Eye. Feet.	x Miles.
1′	1′·7	17′	29′·6	1	1·1	17	4′·4
2	3·5	18	31·4	2	1·5	18	4·5
3	5·2	19	33·1	3	1·8	19	4·7
4	7·0	20	34·9	4	2·1	20	4·8
5	8·7			5	2·4	21	4·9
6	10·5			6	2·6	22	5·0
7	12·2			7	2·8	23	5·1
8	13·9			8	3·0	24	5·2
9	15·7			9	3·2	25	5·4
10	17·4			10	3·4	26	5·5
11	19·2			11	3·6	27	5·6
12	20·9			12	3·7	28	5·7
13	22·7			13	3·9	29	5·8
14	24·4			14	4·0	30	5·9
15	26·2			15	4·1	31	6·0
16	27·9			16	4·2	32	6·1

The foregoing Table IV. will be found useful in calculating the heights from the observed angles of elevation as follows.

From Table I. we find the angle of elevation of P above the high water line at A observed from S_1 is 58′ 20″, hence $m=58{\cdot}33$; S_1P or $d=15'{\cdot}28$, S_1A or $y=2'{\cdot}98$, $x=4{\cdot}54$, inserting these values in equation (3) we have

$$\left.\begin{array}{r}\text{height of } P\\ \text{above sea level}\end{array}\right\}=(58{\cdot}32+7{\cdot}64+1{\cdot}56)26{\cdot}4-18\left(\frac{30{\cdot}56}{2{\cdot}98}-1\right)\text{ft.}$$

$$=67{\cdot}52\times 26{\cdot}4-18\times 9{\cdot}25$$

$$=1782{\cdot}5-166{\cdot}5=1616 \text{ feet.}$$

When the distances of the high water line and the object from the observer are nearly equal, the term $md \times 6000 \sin 1'$ will

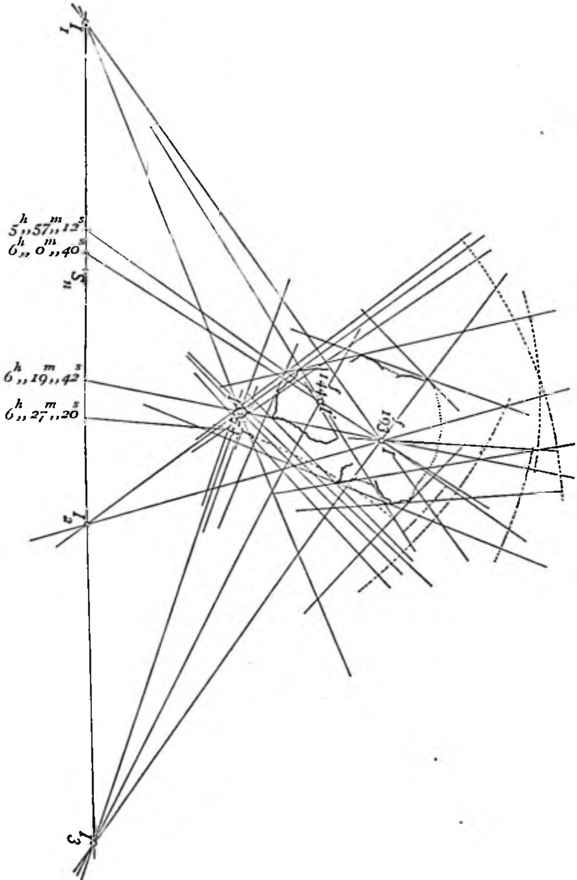


Fig. 79.

give the height with sufficient accuracy; thus from Table I. at chronometer time $12^h\ 31^m\ 45^s$ the altitude of C above its high

water line was $15' \, 35'' = 15'{\cdot}6$; measuring its distance from the ship at that time off the projection we find $d = 1'{\cdot}94$; entering Table IV. with this distance we have $d \times 6000 \sin 1' = 3{\cdot}4$.

$\therefore$ height of C above high water line $= 15{\cdot}6 \times 3{\cdot}4 = 53$ feet.

Also at $1^h \, 2^m \, 14^s$ the altitude of the cliff at c_2 above its high water line was $12' \, 37''{\cdot}5$, the distance of the cliff from the ship at that time being, as measured from the plan, 2·62 miles.

$$\begin{array}{lrl} \text{Entering Table IV. we have} & 2 \text{ miles} & = 3{\cdot}5 \\ & 0{\cdot}6 \;\;,, & = 1{\cdot}05 \\ & 0{\cdot}02 \;,, & = 0{\cdot}03 \\ \hline \therefore & 2{\cdot}62 \;,, & = 4{\cdot}6 \end{array}$$

$\therefore$ height of cliff at c_2 above sea level $= 12{\cdot}6 \times 4{\cdot}6 = 58$ feet.

We will now take the observations made to determine the island. It was necessary to use another sheet prepared and stretched on another board for the purpose (see Fig. 79). In this the scale is one inch to the mile—the straight line $I_1I_2I_3$ representing the ship's course, N. 75° 30′ W., having been carefully drawn in the chart room beforehand. I_1 was taken for the ship's position at chronometer time $5^h \, 26^m \, 48^s$, when the engines were stopped and the observations recorded in Table I. for that time taken. The magnetic bearing of I at that time was S. 58° 30′ W. and the ship's course W. ¼ N.; 34° 19′ was therefore the angle between these two directions. Laying this angle off from I_1 to the left of $I_1I_2I_3$ we have the straight line I_1I passing through I as shown in the figure, and the other lines of direction to the other objects were drawn in a manner similar to that already described from the angles as they were observed in succession, and afterwards from the other positions of the ship in succession, the coast line being sketched in between the points as they were fixed in succession; the appearance of the land was also sketched from positions I_1 and I_3 of the ship as shown in Fig. 76. The rough sketch was afterwards corrected as follows.

$$\begin{array}{lr} \text{Reading of patent logs at} & I_1 = 26'{\cdot}86 \\ ,, \qquad\qquad ,, & I_2 = 32{\cdot}94 \\ ,, \qquad\qquad ,, & I_3 = 36{\cdot}75 \\ \therefore \text{ patent log distance} & I_1I_2 = \;\; 6'{\cdot}08 \\ ,, \qquad\qquad ,, & I_1I_3 = \;\; 9{\cdot}89 \end{array}$$

As before these must be multiplied by 0·956, and by doing so we find

$$\begin{array}{lr} \text{corrected distance} & I_1I_2 = 5'{\cdot}812 \\ ,, \qquad\quad ,, & I_1I_3 = 9{\cdot}454 \end{array}$$

To find the sun's hour angle at I_1 at chronometer time $5^h\,26^m\,48^s$, chronometer gained since sights taken at S_1, $0{\cdot}5^s$; I_1 west of S_1, $2^m\,1^s$.

At chron. time $11^h\,58^m\,23^s$ chron. fast S_1 mean time		$5^h\,53^m\,15^s{\cdot}4$
„ 5 26 48 „ gained „		+ 0 ·5
I_1 west of S_1 - - - - - -		2 1
Chron. fast I_1 mean time at time ⊙ obs. taken		5 55 16 ·9
Mean noon precedes apparent noon - -		5 32 ·8
Chron. time of apparent noon at I_1 at same time		6 0 49 ·7
Chronometer time of observation - - -		5 26 48
Sun's hour angle when its altitude was taken		34 1 ·7

The observed altitude of the sun's lower limb was 69° 43′ 20″ and true altitude ⊙ 69° 54′ 37″, the sun being east of the meridian; to find true bearing

log sin hour angle 34^m 1 ·7	- -	9·1701
log sin Pol. Dist. 73° 30′·7	- -	9·9817
log cosec Zen. Dist. 20 5 ·4	- -	0·4644
log sin true bearing 24 24 ·5	- -	9·6162

1′ error in the sun's hour angle gives 3′ error in the true bearing.
1′ „ „ obsd. alt. „ 1·2 „ „

At I_1 we have ⊙| 91° 22′ I or ⊙ 91° 38 I ⊙ true alt. 69° 55′

log cos 91° 38′	-	8·4549
log sec 69 55	-	0·4643
log cos 94 45·8	-	8·9192

1′ error in observed angle gives 2′·9 error in horizontal angle.

From I_1 true bearing ⊙		S. 24° 24′·5 E.
Horizontal angle ⊙	-	94 45 ·8 I
True bearing I -	-	S. 70 21 ·3 W.
Ship's true course	-	N. 75 31 ·5 W
Angle II_1I_3 -	-	34 7 ·2

At I_2 chronometer time - - - - - -	$6^h\ 41^m\ 3^s$
Gain since time of S_1 sight - - - -	0·7
I_2 west of S_1 - - - - - - -	2 22·3
Chron. fast of S_1 mean time at time of sight -	5 53 15·4
Chron. fast mean time I_2 at time of observation	5 55 38·4
Equation of time - - - - - -	5 32·5
Chron. fast of apparent time at I_2 at chronometer time $6^h\ 41^m\ 3^s$ - - - -	6 1 10·9
⊙ hour angle after noon - - - - -	39 52

log sin $39^m\ 52^s$	- 9·2382	⊙ true altitude		69° 27′·2
log sin 73° 31′	- 9·9818	declination	-	16 29 N.
log cosec 20 33	- 0·4547			
log sin 28 13	- 9·6747			

From Table I., P 76° 37′·2 |⊙ or P 76° 53′ ⊙

log cos 76° 53′ -	9·3559
log sec 69 27 -	0·4547
log cos 49 42·8 -	9·8106

	1′ error in observed angle	gives	3′·6 error in hor. angle.
also	1′ error in the sun's hour angle	„	3′ error in true bearing.
„	1′ „ „ obsd. alt.	„	1′·4 „ „

At I_2 true bearing ⊙ - -	S. 28° 13′ W.
Horizontal angle P - -	49 42·8 ⊙
True bearing of P from I_2 -	S. 21 29·8 E.
Observed angle PI_2I - -	19 40
True bearing of I - -	S. 1 49·8 E.
Ship's back course from I_2 to I_1	S. 75 31·5 E.
Angle I_1I_2I - - -	73 41·7

At I_3 chronometer time $7^h\ 28^m\ 2^s$ we find in a similar manner ⊙ hour angle $1^h\ 26^m\ 30^s{\cdot}4$, true altitude ⊙ 61° 28′, and N.P.D. 73° 31′·9.

log sin hour angle	$1^h\,26^m\,30^s{\cdot}4$	-	9·5665
log sin N.P.D. -	- 73° 32′	-	9·9818
log cosec - -	- 28 32	-	0·3209
log sin true bearing	- 47 44	-	9·8692

At I_3 ... I 88° 41′·2 $\lfloor\odot$ or 88° 57′ $\odot$

log cos 88° 57′	- -	8·2630
log sec 61 28	- -	0·3209
log cos 87 48	- -	8·5839

1′ error in obsd. angle gives 2′·1 error in horizontal angle.
1 „ $\odot$ hour „ „ 2·8 „ sun's true bearing.
1 „ $\odot$ altitude „ 2·0 „ „ „

At I_3 true bearing $\odot$	- -	S. 47° 44′ W.
Horizontal angle	- - -	I 87 48 $\odot$
True bearing I	- - -	S. 40 4 E.
Ship's back course I_3I_1	- -	S. 75 31·5 E.
Angle I_1I_3I	- - - -	35 27·5

For triangle I_1II_3 we have $I_1I_3 = 9'{\cdot}454$ miles;

$$\begin{aligned} \text{angle } II_1I_3 &= 34^\circ\ 7'{\cdot}2 \\ I_1I_3I &= 35\ \ 27{\cdot}5 \\ \therefore\quad I_1II_3 &= 110\ \ 25{\cdot}3 \end{aligned}$$

log 9·454 - - -	0·9756	- - - - -	- 0·9756
log sin 34° 7′·2 -	9·7489	log sin 35° 27′·5 -	- 9·7635
log cosec 110 25·3 -	0·0282	- - - -	- 0·0282
log II_3(=5·659) -	0·7527	log II_1(=5·852) -	- 0·7673

For triangle I_1II_2 we have $I_1I_2 = 5'{\cdot}812$ miles.

$\angle II_1I_3 = 34^\circ\ 7'{\cdot}2$; $\angle I_1I_2I = 73^\circ\ 41'{\cdot}7$; and $\therefore \angle I_1II_2 = 72^\circ\ 11'{\cdot}1$

log 5·812 - -	0·7643	- - - -	- 0·7643
log sin 34° 7′·2 -	9·7489	log sin 73° 41′·7	- 9·9822
log cosec 72° 11′·1 -	0·0213	- - - -	- 0·0213
log II_2(=3′·424) -	0·5345	log II_1(=5′·859)	- 0·7678

For triangle II_2I_3 we have $I_2I_3 = 3'{\cdot}642$ miles.
$\angle II_2I_3 = 106° \; 18'{\cdot}3$; $\angle I_2I_3I = 35° \; 27'{\cdot}5$; and $\therefore \angle I_2II_3 = 38° \; 14'{\cdot}2$

log 3·642	-	0·5613	- - -	-	0·5613
log sin 106° 18′·3	-	9·9822	log sin 35° 27′·5	-	9·7635
log cosec 38° 14′·2	-	0·2084	- - -	-	0·2084
log II_3(=5·648)	-	0·7519	log II_2(=3·414)	-	0·5332

Here we have two values of each of the distances, II_1, II_2, and II_3, which should be meaned according to the values of the two determinations of each respectively. Each of the true bearings of I determined from the observations made at I_1, I_2, and I_3 are liable to errors from three sources—first, an error in the sun's hour angle; second, an error in the observed altitude of the sun; and third, an error in the angle between the sun and I; also the rate of the ship over the ground between I_1 and I_2, when compared with that between I_2 and I_3, may not have been exactly and uniformly equal, so that I_1I_2, instead of being $5'{\cdot}812$ miles, was in fact $5{\cdot}812 + y$ miles, making I_2I_2 $3'{\cdot}642 - y$ miles instead of 3·642 miles. Suppose half the difference between each pair of values of the above distances of I from the three positions of the ship due to this cause, and the other half due to the errors made in the sextant observations remaining uncorrected, and to that of the sun's hour angle determined from the error of the chronometer on S_1 mean time and ship's run from S_1 corrected. We shall have $II_1 = 5{\cdot}852$ and $5{\cdot}859 + \frac{5{\cdot}859}{5{\cdot}812}y$, and if half the difference 0·007 is due to the error y, we have

$$y = -0{\cdot}0035 \ldots\ldots\ldots\ldots (1)$$

also from
$$\left.\begin{aligned} II_2 &= 3{\cdot}414 - \frac{3{\cdot}414}{3{\cdot}642}y, \\ \text{and} \quad &= 3{\cdot}424 + \frac{3{\cdot}424}{5{\cdot}648}y. \end{aligned}\right\}$$

$\therefore$ upon the same supposition

$$1{\cdot}5 \times y = -0{\cdot}005 \ldots\ldots\ldots\ldots (2)$$

Comparing the two values of II_3,

$$\left.\begin{aligned} II_3 &= 5{\cdot}659 \\ \text{also} \quad &= 5{\cdot}648 - \frac{5{\cdot}648}{3{\cdot}642}y, \end{aligned}\right\}$$

which, upon the same supposition, gives

$$1{\cdot}5 \times y = -0{\cdot}0055 \ldots\ldots\ldots\ldots (3)$$

Meaning these values of y in proportion to the coefficients of y we have

$$\begin{aligned} 2y &= -0\cdot007 \\ 4\cdot5y &= -0\cdot015 \\ 4\cdot5y &= -0\cdot0165 \\ \hline 11y &= -0\cdot0385 \\ \therefore\ y &= -0\cdot035 \end{aligned}$$

Applying this value of y, we have

$$\begin{aligned} &II_1 = 5\cdot852 \text{ and } 5\cdot855 \text{ from } && I_1I_2 = 5\cdot8085 \\ &II_2 = 3\cdot4176 \text{ and } 3\cdot4216 \text{ from } && \begin{cases} I_1I_2 = 5\cdot8085 \\ I_2I_3 = 3\cdot6455 \end{cases} \\ &II_3 = 5\cdot659 \text{ and } 5\cdot654 \text{ from } && I_2I_3 = 3\cdot6455 \end{aligned}$$

Observations at I_1,

an error of 1′ in hour angle gave 3′ error T.B.,
an error of 1′ in altitude angle gave 1′·2 error T.B.,
an error of 1′ angle between I and ⊙ gave 2′·9 error T.B.,

and supposing each of the errors to be of same amount and same size, the sum of the errors as regards the size of these effects on the true bearing of I may be expressed by 7·1. In the same manner the comparative size of the effects of the errors of the true bearing of I from I_2 is 8, and of the true bearing of I from I_3 is 6·9; hence if x be the error of the true bearing of I determined from I_1,

$$\begin{aligned} &\text{the angle } II_1I_2 \text{ will} = 34^\circ\ 7'\cdot2 + x, \\ &\text{the angle } I_1I_2I \text{ will} = 73\ \ 41'\cdot7 - \frac{8}{7}x, \\ &\text{the angle } I_1I_3I \text{ will} = 35\ \ 27'\cdot5 - x. \end{aligned}$$

In the triangle I_1II_3 the two angles II_1I_3 and I_1I_3I are changed from 34° 7′·2 to 34° 7′·2$+x$, and from 35° 27′·5 to 35° 27′·5$-x$ respectively.

The change in the logarithm of II_1, arising from these changes, will be due to the change in the log sine of 35° 27′·5 to that of 35° 27′·5$-x$ or $-178x$, and therefore will be diminished by $178x$. The change in the logarithm of II_3, resulting from the change in the angle II_1I_3, will in a similar manner be increased by $186x$, taking the logarithms to two places of figures more than in the calculations, viz., to six places instead of four.

In the triangle I_1II_2, the change in the logarithms of II_1 and II_2 consequent on the above change in the angles will be—

$$
\begin{aligned}
&\text{Alteration in log sin } I_1I_2I = -37\tfrac{8x}{7},\\
&\quad ,, \quad \text{log cosec } I_2II_1 = -41\tfrac{x}{7},\\
&\quad ,, \quad \log II_1 = -48x,\\
&\quad ,, \quad \text{log sin } II_1I_2 = 186x,\\
&\quad ,, \quad \text{cosec } I_2II_1 = -\tfrac{41x}{7},\\
&\quad ,, \quad \log II_2 = 180x.
\end{aligned}
$$

In triangle I_2II_3—

$$
\begin{aligned}
&\text{Alteration in log sin } I_2I_3I = -178x,\\
&\quad ,, \quad \text{log cosec } I_3II_2 = 160\tfrac{x}{7},\\
&\quad ,, \quad \log II_2 = -155x,\\
&\quad ,, \quad \text{log sin } II_2I_3 = -37\tfrac{8x}{7},\\
&\quad ,, \quad \text{log cosec } I_3II_2 = 160\tfrac{x}{7},\\
&\quad ,, \quad \log II_3 = -14x.
\end{aligned}
$$

Triangle I_1II_3 gave $\log II_1 = 0{\cdot}767304$,
corrected for change in angles $= 0{\cdot}767304 - 178x$.

In triangle I_1II_2 $\log II_1$, corrected for change in I_1I_2, $= 0{\cdot}767527 - 48x$.

Therefore equating these two values of II_1,

$$223 + 136x = 0 \quad\text{..........................(1)}$$

In triangle I_1II_2 after first correction

$$II_2 = 3{\cdot}4216, \ \log = 0{\cdot}534229$$
$$\text{log alteration of angles} = 0{\cdot}534229 + 180x.$$

In triangle I_2II_3 after correction

$$II_2 = 3{\cdot}4176, \ \log = 0{\cdot}533721 - 155x.$$

Equating these values of II_2,

$$508 + 335x = 0 \quad\text{..........................(2)}$$

In the same triangle after first correction

$$II_3 = 5{\cdot}654, \quad \therefore \log = 0{\cdot}752356 - 14x.$$

Triangle I_1II_3 gives $II_3 = 0{\cdot}752740$,
and when angles corrected $= 0{\cdot}752740 + 186x$;

$\therefore$ equating these values of II_3 we have

$$384 + 202x = 0 \quad\text{..........................(3)}$$

Multiply (1) by 2, (2) by 5, and (3) by 3, and add,

$$\begin{aligned} 446+\ \ 260x&=0\\ 2540+1675x&=0\\ 1152+\ \ 606x&=0\\ \hline 4138+2541x&=0 \end{aligned}$$

$$\therefore \qquad x=-\frac{4138}{2541}=-1'\cdot 6$$

$$\begin{aligned} \therefore \qquad II_1I_2&=34^\circ\ \ 7'\cdot 2-1'\cdot 6=34^\circ\ \ 5'\cdot 6\\ I_1I_2I&=73\ \ 41\cdot 7+1\cdot 8=73\ \ 43\cdot 5\\ I_1I_3I&=35\ \ 27\cdot 5+1\cdot 6=35\ \ 29\cdot 1. \end{aligned}$$

Adding, we have

$$2\log II_1=1\cdot 534831-226x,$$
$$2\log II_1=0\cdot 767416+181=0\cdot 767597;$$

$$\therefore \qquad II_1=5'\cdot 856.$$

Similarly $\quad 2\log II_2=1\cdot 067950+25x;$

$$\therefore \qquad \log II_2=0\cdot 533975-20=0\cdot 533955;$$

$$\therefore \qquad II_2=3'\cdot 419;$$

also $\quad 2\log II_3=1\cdot 505096+172x,$

$$\log II_3=0\cdot 752548-138=0\cdot 752410;$$

$$\therefore \qquad II_3=5\cdot 655.$$

To make the fair projection—Draw the straight line $I_1I_2I_3$ right across a sheet of paper laid on the chart table, and between 1 and 2 inches from its lower edge (see Fig. 80, where the scale of projection is one inch to a mile, though in practice it is far better to adopt as large a scale as the sheet will allow), make $I_1I_3=9'\cdot 454$ miles, and $I_1I_2=5'\cdot 8085$; also $I_1I=5'\cdot 856$, $I_2I=3'\cdot 419$, and $I_3I=5'\cdot 655$. After I is projected it is used as the point of reference from which all the angles observed from the ship in her several positions must be laid off, and in this manner the positions of the other points were determined, and the coast line between them sketched in from the rough sheet.

To determine the position of S_n between I_1 and I_2 at chronometer time $6^h\ 1^m\ 12^s$, when the patent logs were read and the sun's meridian altitude determined from observations, the mean reading of the patent logs was $29'\cdot 70$, the reading at I_1 was $26'\cdot 86$ and at I_2 $32'\cdot 94$; $\therefore$ patent log distance $I_1I_2=6'\cdot 08$;

corrected distance $I_1I_2 = 5'{\cdot}8085$. Chronometer time at I_1 was $5^h\ 26^m\ 48^s$, that at I_2 was $6^h\ 41^m\ 3^s$. The ship made good

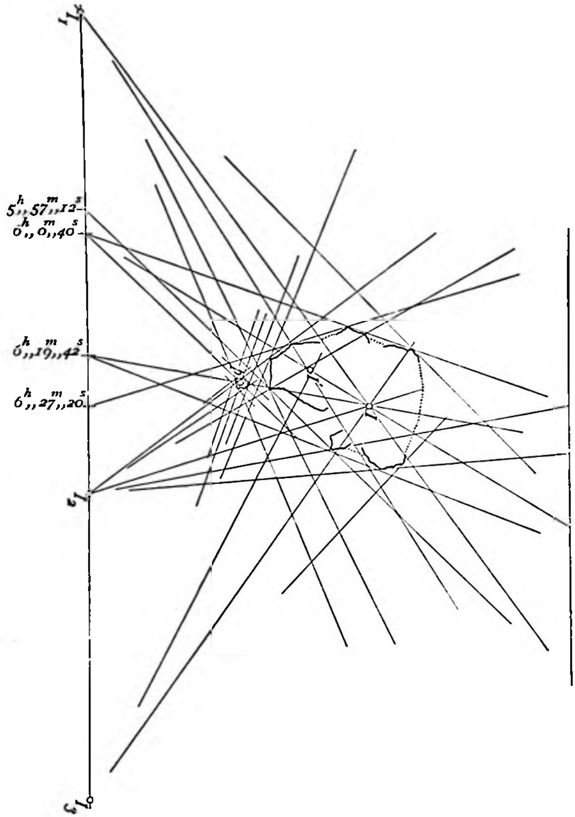


FIG. 80.

$5'{\cdot}8085$ in the interval between these times, or in $74^m\ 15^s$. From I_1 the ship ran at same rate for $34^m\ 24^s$; therefore distance $S_1S_n = 5'{\cdot}8085\dfrac{34{\cdot}4}{74{\cdot}25} = 2'{\cdot}691$.

log 5·8085	- - - - -	0·764064
log 34·4	- - - - -	1·536558
		2·300622
log 74·25	- - - - -	1·870696
log 2·691	- - - - -	0·429926

By patent log readings $I_1S_n = 2·84$;

$$\therefore \qquad I_1S_n \text{ corrected} = 2'{\cdot}84\frac{5{\cdot}8085}{6{\cdot}0800} = 2'{\cdot}713, \text{ for}$$

log 5·8085	- - - -	0·7641
log 6·08	- - - -	0·7839
		$\bar{1}$.9802
log 2·84	- - - -	0·4533
log I_1S_n corrected (2·713)	-	0·4335

Comparing this with I_1S_n determined by time we find a sufficiently near agreement, when we consider that the occasional stopping of the engines slightly deranged the uniformity of the ship's rate.

The other positions of the ship at the times given by the chronometer were determined in the same manner by means of the time intervals.

For the height of the top of the pinnacle rock P above the sea level, we observe that P and its high water line are so nearly equidistant from the ship that we may so consider them.

At chronometer time $6^h\ 19^m\ 42^s$ angle of elevation of P above its high water mark was 25′ 17″·5. At this time its distance from the ship was 1′·76 miles. (See Fig. 80.)

From Table IV. - - 1′ $= 1·7$,

0·7 $= 1·22$,

0·06 $= 0·11$;

$\therefore$ 1·76 $= 3·03$;

$\therefore$ height of $P = 3·03 \times 25·3 = 76·6$ feet.

At I_n the angle of elevation of P above its high water mark was 20′ 20″, and its distance from P 2′·16 (miles).

Table IV. gives - - 2′ $= 3·5$

0·1 $= 0·17$

0·06 $= 0·11$

2·16 $= 3·78$

Height of $P = 3{\cdot}78 \times 20{\cdot}33 = 76{\cdot}85$ feet.
P's elevation may therefore be taken as 77 feet.

At chronometer time $6^h\ 19^m\ 42^s$ the angle of elevation of I above the high water line of the pinnacle rock was $37'\ 50''$ and $I'\ 33'\ 45''$.

Taken from plan, ship $I = 3'{\cdot}33$, ship P, high water line $1'{\cdot}76$.
ship $I' = 2{\cdot}6$, height of eye 18 feet.

$\therefore x = 4'{\cdot}54$ and $y = 1'{\cdot}76$ in first case and $2'{\cdot}13$ in the second.

for I we have $m + \frac{d}{2} + x - y = 37{\cdot}83 + 1{\cdot}665 + 4{\cdot}54 - 1{\cdot}76$
$= 42{\cdot}27$

for I' „ $m + \frac{d}{2} + x - y = 33{\cdot}75 + 1{\cdot}3 + 4{\cdot}54 - 2{\cdot}13$
$= 37{\cdot}5$

Table IV. gives -	$3' = 5{\cdot}2$	Table IV. gives -	$2' = 3{\cdot}5$
	$0{\cdot}3 = {\cdot}52$	- - - -	$0{\cdot}6 = 1{\cdot}05$
	$0{\cdot}03 = {\cdot}052$		$2{\cdot}6 = 4{\cdot}55$
	$3{\cdot}33 = 5{\cdot}77$		

We have now to determine the value of $h\left(\frac{2d}{y} - 1\right)$ and subtract it from the respective products of the quantities above determined.

For the height of I we have $18\left(\frac{6{\cdot}66}{1{\cdot}76} - 1\right) = 2{\cdot}78 \times 18$
$= 50$

For the height of I' we have $18\left(\frac{5{\cdot}2}{2{\cdot}13} - 1\right) = 1{\cdot}44 \times 18$
$= 25{\cdot}9$

$\therefore$ height of $I = 42{\cdot}27 \times 5{\cdot}77 - 50 = 243{\cdot}9 - 50 = 194$ feet.
$I' = 37{\cdot}5 \times 4{\cdot}55 - 23{\cdot}5 = 170{\cdot}6 - 25{\cdot}9 = 145$ feet.

At I_2 the observed elevations of I and I_1 above the high water line of the coast immediately below them were respectively $32'\ 42''$ and $28'\ 35''$, height of eye 18 feet. Here the distances of I and the high water mark below it from the ship do not differ much, the distance being $3'{\cdot}4$.

Table IV. gives - - - $3 = 5{\cdot}2$
${\cdot}4 = 0{\cdot}7$

$\therefore 3{\cdot}4 = 5{\cdot}9$

height of $I = 5{\cdot}9 \times 32{\cdot}7 = 193$ feet.

At I_2 we find $I_2I' = 2'{\cdot}96$ and I_2 high water mark below $I' = 2'{\cdot}43$, which gives in same manner as above

height of I' above high water level $= 166{\cdot}7 - 25{\cdot}7 = 141$ feet.

We have now extracted from the day's work sufficient to show how the work ought to be done, and how to use the observations and project the work. After all the observations at S_8 were taken, in addition to the sun's altitude at chronometer time $12^h\ 4^m\ 49^s$, the work for the day was closed and the ship taken to the harbour, for which she was running. There a tide pole was erected at low water and observations made for latitude, and the following day equal altitudes of the sun for time were observed. The calculations of the preceding day's work were commenced and continued until finished and the fair projection made. In the meantime a survey of the harbour was commenced. Before returning to the first harbour equal altitudes of the sun should be observed the day previous to making the run back; if possible a party should be left to observe the tides and continue the survey of the harbour, the tides being the more important, so that as many comparisons of the tides at the two harbours as possible may be secured. The return course should be as nearly as possible parallel to the first, separated from it by at least a mile, but this must depend on the nature of the coast and the distance of the first from the land. The positions of the ship at which the observations of the sun for time are taken should be as nearly as possible given by two hours' run between each, and the rate of the ship kept uniform, such that starting on a line of bearing (magnetic) through S_8 of N. $\frac{1}{4}$ E. and finishing on the same line of bearing P on A as S_1 (the point from which the vessel started on her outward run) is.

The run back should always be made as soon as possible after the first, in order that the tide, the weather, current, and other circumstances tending to derange the course and distance made by the vessel may in both cases be as nearly as possible the same. By this means the combination and comparison of the two runs will eliminate to that extent these errors, and enable their probable effects to be calculated.

END.

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